15-453 Formal Languages, Automata, and Computation

Midterm II

Sample Solution

March 24, 2000

Name: ________________________________

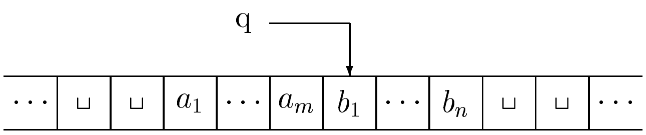
Andrew ID: ________________________________

- This is a closed book exam; only 1 sheet of notes is permitted.
- Write your answer legibly in the space provided.
- You have 50 minutes for this exam.
- This exam consists of 3 questions worth a total of 100 points, plus an extra credit question worth 40 points.
- Extra credit is recorded separately, so make sure to solve Problems 1–3 completely before moving on to extra credit.

Problem 1	Problem 2	Problem 3	Total	Extra Credit
25	35	40	100	40

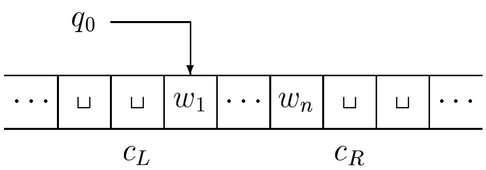
Problem 1 (25 pts)

A Turing machine with a 2-way infinite tape (TM2) is like a regular Turing machine, except that we think of a configuration $a_1 \cdots a_m q b_1 \cdots b_n$ as representing the tape



The initial configuration is still $q_0 w$ for the initial state q_0 and input word w, but the transition relation at the left end of the current configuration works symmetrically to the one at the right end, filling in blanks ⊔ as needed.

We say a machine *uses a cell* if the read/write head points to the cell at least once during the computation of a machine. We use c_L to denote the cell to the left of the input (initially containing ⊔) and c_R the cell to the right of the input (also initially containing ⊔)



For each of the following questions about a TM2 M and input w, check **all** boxes that apply. You do not need to justify your answers.

		decid-able	recogniz-able	co-recogniz-able
a.	M uses c_L		X	
b.	M uses c_R		X	
c.	M uses c_L or c_R	X	X	X
d.	M uses only a finite number of cells		X	
e.	M uses an infinite number of cells			X

Briefly, (a) and (b) are recognizable because we simulate the behavior of M on w and accept if M uses c_L or c_R. They are not decidable by a reduction from the halting problem (omitted here). (c) is decidable since there are only finitely many configurations if neither c_L nor c_R is ever used. Therefore it is also recognizable and co-recognizable. (d) is recognizable because we can simulate M on w and accept if either M halts or repeats a configuration. It is not decidable by a reduction from the halting problem (omitted here). (e) is co-recognizable since its complement (d) is recognizable.

Problem 2 (35 pts)

Prove carefully that the acceptance problem

$$A_{TM2} = \{\langle M, w \rangle \mid M \text{ is a TM2 accepting } w\}$$

is undecidable. You may use a mapping reduction or a proof by contradiction. In either case you should clearly state your proof method and explain your construction in detail.

We assume that A_{TM2} is decided by a TM D and derive contradiction. We do this by constructing a TM S which decides the acceptance problems for Turing machines A_{TM}, which is impossible, since A_{TM} is undecidable.

S = "On input $\langle M, w \rangle$

1. Construct the representation of a TM2 T which is like M, except that it first marks the square to the left of the input and immediately moves right again if this mark is seen in any state.
2. Run D on $\langle T, w \rangle$ and

a. accept if D accepts,

b. reject if D rejects."

S always halts, since D decides A_{TM2}. Furthermore, S accepts $\langle M, w \rangle$ if and only if M accepts w, and it rejects $\langle M, w \rangle$ if and only if M rejects w. It therefore solve A_{TM} which is a contradiction.

Problem 3 (40 pts)

a. Is the acceptance problem for context-free grammars

$$A_{CFG} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$$

decidable? You only need to answer yes or no (this was covered in class).

Yes

b. Is the problem

$$ALL_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG with } \mathcal{L}(G) = \Sigma^*\}$$

decidable? You only need to answer yes or no (this was covered in class).

No

c. Prove that

$$EQ_{CFG} = \{\langle G_1, G_2\rangle \mid G_1 \text{ and } G_2 \text{ are CFGs with } \mathcal{L}(G_1) = \mathcal{L}(G_2)\}$$

is undecidable. You may use the answers to (a) and (b).

We give a mapping reduction from ALL_{CFG} which is undecidable (part (b)). Let G_Σ be a grammar which generates every word $w \in \Sigma^*$. The reduction f maps the representation of a grammar $\langle G\rangle$ to the pair $\langle G, G_\Sigma\rangle$. Clearly, $\langle G\rangle \in ALL_{CFG}$ if and only if $\mathcal{L}(G) = \Sigma^* = \mathcal{L}(G_\Sigma)$, so f is a correct reduction. It is computable, since G_Σ is easily constructed from Σ.

d. Prove that EQ_{CFG} is co-Turing-recognizable. You may use the answers to (a) and (b).

The negation of EQ_{CFG} contains the grammars which generate different languages, plus the strings which are not valid pairs of grammars. The latter is easy to recognize. For the former, we assume the input has the form $\langle G_1, G_2 \rangle$. If G_1 and G_2 generate different languages, then there must a word generated by one but not the other. We therefore enumerate the words w in Σ^* and accept if $\langle G_1, w \rangle$ is in A_{CFG} and $\langle G_2, w \rangle$ is not, or vice versa. This property is decidable by A_{CFG} (part (a)).

Problem 4 (40 pts extra credit)

A *context-sensitive grammar* (CSG) $C = (V, \Sigma, R, S)$ is like a context-free grammar, except that the rules may have the form

$$xAy \longrightarrow w$$

for $x, y \in \Sigma_\epsilon$, $A \in V$, and $w \in (V \cup \Sigma)^*$. We permit a derivation step $uxAyv \Longrightarrow uwv$ if $xAy \longrightarrow w$ is in R. Note that x and y may either be single letters in Σ or empty, so a context-free grammar is a special case of a context-sensitive grammar. As before

$$\mathcal{L}(C) = \{w \mid S \Longrightarrow \cdots \Longrightarrow w \text{ for } w \in \Sigma^*\}$$

Prove that the emptiness problem for context-sensitive grammars

$$E_{CSG} = \{\langle C\rangle \mid C \text{ is a CSG and } \mathcal{L}(C) = \emptyset\}$$

is undecidable by a reduction from the acceptance problem for Turing machines with two-way infinite tapes, A_{TM2}.

Hint: Represent states of the Turing machine by non-terminals and configurations by strings.

We show the reduction. Given a TM2 $M = (Q, \Sigma, \Gamma, \delta, q_0, q_A, q_R)$ and input word w we have to construct a CSG $C = (V, \Gamma, R, S)$ such that M accepts w iff $S \Longrightarrow \cdots \Longrightarrow u$ for some $u \in \Gamma^*$.

As indicated, the set of terminals of C is Γ, the tape alphabet of M.

We choose $V = Q \cup \{S, B\}$, where S is the start symbol and B generates an arbitrary number of blanks.

1. The production modeling the initial configuration is

$$S \longrightarrow Bq_0wB$$

2. The productions modelig the blank tape

$$B \longrightarrow \sqcup B \mid B\sqcup \mid \epsilon$$

3. The productions modeling right-moving transitions $\delta(q, a) = (\hat{q}, b, R)$

$$qa \longrightarrow b\hat{q}$$

4. The productions modeling left-moving transitions. $\delta(q, a) = (\hat{q}, b, L)$.

$$cqa \longrightarrow \hat{q}cb$$

for any $c \in \Gamma$.

5. The production(s) modeling acceptance.

$$q_A \longrightarrow \epsilon$$

6. The production(s) modeling rejection.

no production for q_R