3.5 Applications of Cut Elimination

The cut elimination theorem is the final piece needed to complete our study of natural deduction and normal natural deduction and at the same time the springboard to the development of efficient theorem proving procedures. Our proof in the previous section is constructive and therefore contains an algorithm for cut elimination. Because the cases are not mutually exclusive, the algorithm is non-deterministic. However, the resulting derivation should always be the same. While this property does not quite hold, the different derivations can be shown to be equivalent in a natural sense. This is called the *confluence* property for intuitionistic cut elimination modulo commutative conversions. It it is not implicit in our proof, but has to be established separately.[2] On the other hand, our proof shows that any possible execution of the cut-elimination algorithm terminates. This is called the *strong normalization* property for the sequent calculus.

By putting the major results of this chapter together we can now prove the normalization theorem for natural deduction.

Theorem 3.13 (Normalization for Natural Deduction)
If $\Gamma \vdash A$ *then* $\Gamma^{\downarrow} \vdash A \Uparrow$.

Proof: Direct from previous theorems.

$\Gamma \vdash A$	Assumption
$\Gamma^{\downarrow} \vdash^{+} A \Uparrow$	By completeness of annotated deductions (Theorem 3.3)
$\Gamma \stackrel{+}{\Longrightarrow} A$	By completeness of sequent calculus with cut (Theorem 3.10)
$\Gamma \Longrightarrow A$	By cut elimination (Theorem 3.12)
$\Gamma^{\downarrow} \vdash A \Uparrow$	By soundness of sequent calculus (Theorem 3.6)

□

Among the other consequences of cut elimination are consistency and various independence results.

Corollary 3.14 (Consistency) *There is no deduction of* $\vdash \bot$.

Proof: Assume there is a deduction $\vdash \bot$. By the results of this chapter then $\cdot \Longrightarrow \bot$. However, this sequent cannot be the conclusion of any inference rule in the (cut-free) sequent calculus. Therefore $\vdash \bot$ cannot be derivable. □

In the same category are the following two properties. As in the proof above, we analyze the inference rules which may have led to a given conclusion. This proof technique is called *inversion*.

Corollary 3.15 (Disjunction and Existential Property)

1. *If* $\vdash A \vee B$ *then either* $\vdash A$ *or* $\vdash B$.

[2] [*reference?*]

2. If $\vdash \exists x.\ A$ then $\vdash [t/x]A$ for some t.

Proof: Direct by inversion on possible sequent derivations in both cases.

1. Assume $\vdash A \vee B$. Then $\cdot \Longrightarrow A \vee B$. By inversion, either $\cdot \Longrightarrow A$ or $\cdot \Longrightarrow B$. Therefore $\vdash A$ or $\vdash B$.

2. Assume $\exists x.\ A$. then $\cdot \Longrightarrow \exists x.\ A$. By inversion, $\cdot \Longrightarrow [t/x]A$ for some t. Hence $\vdash [t/x]A$.

□

Note that the disjunction and existential properties rely on a judgment without hypotheses. For example, we have $B \vee A \Longrightarrow A \vee B$, but neither $B \vee A \Longrightarrow A$ for $B \vee A \Longrightarrow B$ hold.

The second class of properties are *independence* results which demonstrate that certain judgments are not derivable. As a rule, these are parametric judgments some instances of which may be derivable. For example, we will show that the law of excluded middle is independent. Nonetheless, there are some propositions A for which we can show $\vdash A \vee \neg A$ (for example, take $A = \bot$).

Corollary 3.16 (Independence of Excluded Middle)
There is no deduction of $\vdash A \vee \neg A$ for arbitrary A.

Proof: Assume there is a deduction of $\vdash A \vee \neg A$. By the result of this section then $\cdot \Longrightarrow A \vee \neg A$. By inversion now either $\cdot \Longrightarrow A$ or $\cdot \Longrightarrow \neg A$. The former judgment (which is parametric in A) has no derivation. By inversion, the latter can only be infered from $A \Longrightarrow p$ for a new parameter p. But there is no inference rule with this conclusion, and hence there cannot be a deduction of $\vdash A \vee \neg A$. □

3.6 Proof Terms for Sequent Derivations

In this section we address the question of how to assign proof terms to sequent calculus derivations. There are essentially two possibilities: we can either develop a new proof term calculus specifically for sequent derivations, or we can directly assign natural deduction proof terms. The former approach can be found, for example, in [Pfe95]. The latter is more appropriate for our purposes here, since we view natural deductions as defining truth and since we already devised methods for compact representations in Section 3.2.

We define a new judgment, $\Gamma \Longrightarrow I : A$, maintaining that $\Gamma \vdash I : A$. For this purpose we abandon the previous convention of omitting labels for hypotheses, since proof terms need to refer to them. On the other hand, we still consider assumptions modulo permutations in order to simplify notation. We use the compact proof terms here only for simplicity.

The proof terms to be assigned to each inference rule can be determined by a close examination of the soundness proof for the sequent calculus (Theorem 3.6).

Since that proof is constructive, it contains an algorithm for translating a sequent derivation to a normal natural deduction. We just have to write down the corresponding proof terms.

Initial Sequents. These are straightforward.

$$\frac{}{\Gamma, u{:}A \Longrightarrow u : A}\,\text{init}$$

Note that there may be several hypotheses A with different labels. In the shorthand notation without labels before, it is ambiguous which one was used.

Conjunction. The right rule is straightforward, since it is isomorphic to the introduction rule for natural deduction. The left rules require a substitution to be carried out, just as in the proof of Theorem 3.6.

$$\frac{\Gamma \Longrightarrow I : A \qquad \Gamma \Longrightarrow J : B}{\Gamma \Longrightarrow \langle I, J\rangle : A \wedge B}\,\wedge\text{R}$$

$$\frac{\Gamma, u{:}A \wedge B, w{:}A \Longrightarrow I : C}{\Gamma, u{:}A \wedge B \Longrightarrow [\text{fst}\, u/w]I : C}\,\wedge\text{L}_1 \qquad \frac{\Gamma, u{:}A \wedge B, w{:}B \Longrightarrow I : C}{\Gamma, u{:}A \wedge B \Longrightarrow [\text{snd}\, u/w]I : C}\,\wedge\text{L}_2$$

There are two potential efficiency problems in the proof term assignment for the left rule. The first is that if w is used many times in I, then $\text{fst}\, u$ or $\text{snd}\, u$ may be replicated many times, leading to a large proof. The second is that when a number of successive left rules are encountered, the term I we substitute into will be traversed many times. These problems can be avoided in several ways (see Exercise **??**).

Implication. The pattern of the previous right and left rules continues here.

$$\frac{\Gamma, u{:}A \Longrightarrow I : B}{\Gamma \Longrightarrow \lambda u.\, I : A \supset B}\,\supset\text{R}$$

$$\frac{\Gamma, u{:}A \supset B \Longrightarrow J : A \qquad \Gamma, u{:}A \supset B, w{:}B \Longrightarrow I : C}{\Gamma, u{:}A \supset B \Longrightarrow [u\, J/w]I : C}\,\supset\text{L}$$

Disjunction. This introduces no new considerations.

$$\frac{\Gamma \Longrightarrow I : A}{\Gamma \Longrightarrow \text{inl}\, I : A \vee B}\,\vee\text{R}_1 \qquad \frac{\Gamma \Longrightarrow J : B}{\Gamma \Longrightarrow \text{inr}\, J : A \vee B}\,\vee\text{R}_2$$

$$\frac{\Gamma, u{:}A \vee B, v{:}A \Longrightarrow I : C \qquad \Gamma, u{:}A \vee B, w{:}B \Longrightarrow J : C}{\Gamma, u{:}A \vee B \Longrightarrow (\,\textbf{case}\ u\ \textbf{of}\ \text{inl}\, v \Rightarrow I \mid \text{inr}\, w \Rightarrow J) : C}\,\vee\text{L}$$

Negation. This is similar to implication.[3]

$$\frac{\Gamma, u{:}A \Longrightarrow I : p}{\Gamma \Longrightarrow \mu^p u.\, I : \neg A}\,\neg\mathrm{R}^p \qquad \frac{\Gamma, u{:}\neg A \Longrightarrow I : A}{\Gamma, u{:}\neg A \Longrightarrow u \cdot I : C}\,\neg\mathrm{L}$$

Truth. This is trivial, since there is no left rule.

$$\frac{}{\Gamma \Longrightarrow \langle\,\rangle : \top}\,\top\mathrm{R}$$

Falsehood. Again, this is immediate.

$$\frac{}{\Gamma, u{:}\bot \Longrightarrow \mathrm{abort} u : C}\,\bot\mathrm{L}$$

To treat the quantifiers we extend our proof term calculus to handle the quantifier rules.[4] We overload the notation by reusing λ-abstraction and pairing. There is no ambiguity, because the proof term for universal quantification binds a term variable x (rather than a proof variable u), and the first component of the pair for existential quantification is a first-order term, rather than a proof term as for conjunction.

First, we show the assignment of these terms to natural deductions, then to the sequent calculus.

Universal Quantification. The proof term for a universal quantifier $\forall x.\, A$ is a function from a term t to a proof of $[t/x]A$. The elimination term applies this function.

$$\frac{\Gamma \vdash [a/x]M : [a/x]A}{\Gamma \vdash \lambda x.\, M : \forall x.\, A}\,\forall\mathrm{I}^a$$

$$\frac{\Gamma \vdash M : \forall x.\, A}{\Gamma \vdash M\, t : [t/x]A}\,\forall\mathrm{E}$$

The local reductions and expansions just mirror the corresponding operations on natural deductions.

$$\begin{array}{lll} (\lambda x.\, M)\, t & \longrightarrow_R & [t/x]M \\ M : \forall x.\, A & \longrightarrow_E & \lambda x.\, M\, x \quad (x \text{ not free in } M) \end{array}$$

Existential Quantification. The proof term for an existential $\exists x.\, A$ is a pair consisting of a witness term t and the proof of $[t/x]A$.

$$\frac{\Gamma \vdash M : [t/x]A}{\Gamma \vdash \langle t, M\rangle : \exists x.\, A}\,\exists\mathrm{I}$$

$$\frac{\Gamma \vdash M : \exists x.\, A \qquad \Gamma, u{:}[a/x]A \vdash [a/x]N : C}{\Gamma \vdash \mathbf{let}\ \langle x, u\rangle = M\ \mathbf{in}\ N : C}\,\exists\mathrm{E}^{a,u}$$

[3] *[add to compact proof term section]*
[4] *[move earlier]*

The local reduction for the existential quantifier has to perform two substitutions, just as on natural deductions.

$$\begin{array}{rcl} \mathbf{let}\ \langle x,u\rangle = \langle t,M\rangle\ \mathbf{in}\ N & \longrightarrow_R & [M/u][t/x]N \\ M : \exists x.\, A & \longrightarrow_E & \mathbf{let}\ \langle x,u\rangle = M\ \mathbf{in}\ \langle x,u\rangle \end{array}$$

It is once again easy to see how to divide the proof terms into introduction and elimination forms. We only show the resulting definition of compact proof terms.

$$\begin{array}{llll} \text{Intro Terms} & I & ::= & \ldots \\ & & & \mid \lambda x.\, I \qquad\qquad \text{Universal Quantification} \\ & & & \mid \langle t,I\rangle \qquad\qquad \text{Existential Quantification} \\ & & & \mid \mathbf{let}\ \langle x,u\rangle = E\ \mathbf{in}\ I \\ \text{Elim Terms} & E & ::= & \ldots \mid E\,t \qquad\qquad \text{Universal Quantification} \end{array}$$

On sequent calculus derivations, we follow the same strategy as in the preceding propositional rules.

Universal Quantification.

$$\frac{\Gamma \Longrightarrow [a/x]I : [a/x]A}{\Gamma \Longrightarrow \lambda x.\, I : \forall x.\, A}\,\forall\mathrm{R}^a \qquad \frac{\Gamma, u{:}\forall x.\, A, w{:}[t/x]A \Longrightarrow I : C}{\Gamma, u{:}\forall x.\, A \Longrightarrow [u\,t/w]I : C}\,\forall\mathrm{L}$$

Existential Quantification.

$$\frac{\Gamma \Longrightarrow I : [t/x]A}{\Gamma \Longrightarrow \langle t,I\rangle : \exists x.\, A}\,\exists\mathrm{R} \qquad \frac{\Gamma, u{:}\exists x.\, A, w{:}[a/x]A \Longrightarrow [a/x]I : C}{\Gamma, u{:}\exists x.\, A \Longrightarrow (\mathbf{let}\ \langle x,w\rangle = u\ \mathbf{in}\ I) : C}\,\exists\mathrm{L}^a$$

3.7 Exercises

Exercise 3.1 Consider a system of normal deduction where the elimination rules for disjunction and existential are allowed to end in an extraction judgment.

$$\frac{\Gamma^{\downarrow} \vdash A \vee B\downarrow \qquad \Gamma^{\downarrow}, u{:}A\downarrow\ \vdash C\downarrow \qquad \Gamma^{\downarrow}, w{:}B\downarrow\ \vdash C\downarrow}{\Gamma^{\downarrow} \vdash C\downarrow}\,\vee\mathrm{E}^{u,w}$$

$$\frac{\Gamma^{\downarrow} \vdash \exists x.\, A\downarrow \qquad \Gamma^{\downarrow}, u{:}[a/x]A\downarrow\ \vdash C\downarrow}{\Gamma^{\downarrow} \vdash C\downarrow}\,\exists\mathrm{E}^{a,u}$$

Discuss the relative merits of allowing or disallowing these rules and show how they impact the subsequent development in this Chapter (in particular, bi-directional type-checking and the relationship to the sequent calculus).

Exercise 3.2

1. Give an example of a natural deduction which is *not* normal (in the sense defined in Section 3.1), yet contains no subderivation which can be locally reduced.

2. Generalizing from the example, devise additional rules of reduction so that any natural deduction which is not normal can be reduced. You should introduce no more and no fewer rules than you need for this purpose.

3. Prove that your rules satisfy the specification in part (2).

Exercise 3.3 Write out the rules defining the judgments $\Gamma^{\downarrow} \vdash^{+} I : A \Uparrow$ and $\Gamma^{\downarrow} \vdash^{+} E : A \downarrow$ and prove Theorem 3.4. Make sure to carefully state the induction hypothesis (if it is different from the statement of the theorem) and consider all the cases.

Exercise 3.4 Fill in the missing subcases in the proof of the admissibility of cut (Theorem 3.11) where A is the principal formula in both $\mathcal{D}$ and $\mathcal{E}$.

Exercise 3.5 Consider an extension of intuitionistic logic by a universal quantifier over propositions, written as $\forall^2 p.\ A$, where p is variable ranging over propositions.

1. Show introduction and elimination rules for $\forall^2$.

2. Extend the calculus of normal and extraction derivations.

3. Show left and right rules of the sequent calulus for $\forall^2$.

4. Extend the proofs of soundness and completeness for the sequent calculus and sequent calculus with cut to accomodate the new rules.

5. Point out why the proof for admissibility of cut does not extend to this logic.

Exercise 3.6 Gentzen's original formulation of the sequent calculus for intuitionistic logic permitted the right-hand side to be empty. The introduction rule for negation then has the form

$$\frac{\Gamma, A \Longrightarrow}{\Gamma \Longrightarrow \neg A}\neg\mathrm{R}.$$

Write down the corresponding left rule and detail the changes in the proof for admissibility of cut. Can you explain sequents with empty right-hand sides as judgments?

Exercise 3.7 The algorithm for cut elimination implicit in the proof for admissibility of cut can be described as a set of reduction rules on sequent derivations containing cut.

1. Write out all reduction rules on the fragment containing only implication.

2. Show the extracted proof term before and after each reduction.
3. If possible, formulate a strategy of reduction on proof terms for natural deduction which directly models cut elimination under our translation.
4. Either formulate and prove a theorem about the connection of the strategies for cut elimination and reduction, or show by example why such a connection is difficult or impossible.

Exercise 3.8

1. Prove that we can restrict initial sequents in the sequent calculus to have the form $\Gamma, P \Longrightarrow P$ where P is an atomic proposition without losing completeness.
2. Determine the corresponding restriction in normal and extraction derivations and prove that they preserve completeness.
3. If you see a relationship between these properties and local reductions or expansions, explain. If you can cast it in the form of a theorem, do so and prove it.

Exercise 3.9 For each of the following propositions, prove that they are derivable in classical logic using the law of excluded middle. Furthermore, prove that they are not true in intuitionistic logic for arbitrary A, B, and C.

1. $((A \supset B) \supset A) \supset A$.
2. Any entailment in Exercise 2.8 which is only classically, but not intuitionistically true.