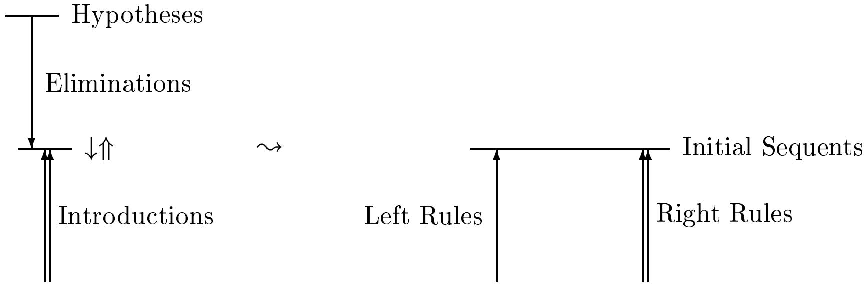
3.3 Sequent Calculus

In Section 3.1 we introduced normal deductions which embody the strategy that proof search should proceed only bottom-up via introduction rules and top-down via elimination rules. The bi-directional nature of this calculus makes it somewhat unwieldy when it comes to the study of meta-theoretic properties and, in particular, complicates its completeness proof. In this section we develop a closely related calculus in which all proof search steps proceed bottom-up. Pictorially, we would like to flip the elimination rules upside-down.



This transformation turns introduction rules into so-called right rules, and upside-down elimination rules into so-called left rules. We have two judgments, A *left* (A is a proposition on the left) and A *right* (A is a proposition on the right). They are assembled into the form of a hypothetical judgment

$$u_1{:}A_1\ \mathit{left}, \ldots, u_n{:}A_n\ \mathit{left} \vdash A\ \mathit{right}.$$

We call such a hypothetical judgment a *sequent*.

Note that the proposition A on the right directly corresponds to the proposition whose truth is established by a natural deduction. On the other hand, propositions on the left do *not* directly correspond to hypotheses in natural deduction, since in general they include hypotheses and propositions derived from them by elimination rules.

Keeping this intuition in mind, the inference rules for sequents can now be constructed mechanically from the rules for normal and extracting derivations. To simplify the notation, we denote the sequent above by

$$A_1, \ldots, A_n \Longrightarrow A$$

where the judgments *left* and *right* are implied by the position of the propositions. Moreover, labels u_i are suppressed until we introduce proof terms. Finally, left rules may be applied to any left proposition. Since the order of the left propositions is irrelevant, we write Γ, A instead of the more pedantic Γ, A, Γ'.

Initial Sequents. These correspond to the coercion from extraction to normal derivations, and *not* to the use of hypotheses in natural deductions.

$$\frac{}{\Gamma, A \Longrightarrow A}\ \mathrm{init}$$

Conjunction. The right and left rules are straightforward and provide a simple illustration of the translation, in particular in the way the elimination rules are turned upside-down.

$$\frac{\Gamma \Longrightarrow A \qquad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\wedge\mathrm{R}$$

$$\frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\wedge\mathrm{L}_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\wedge\mathrm{L}_2$$

In the introduction rule (read bottom-up), we propagate Γ to both premises. This reflects that in natural deduction we can use any available assumption freely in both subdeductions. Furthermore, in the elimination rule the hypothesis $A \wedge B$ *left* persists. This reflects that assumptions in natural deduction may be used more than once. Later we analyze which of these hypotheses are actually needed and eliminate some redundant ones. For now, however, they are useful because they allow us to give a very direct translation to and from normal natural deductions.

Implication. The right rule for implication is straightforward. The left rule requires some thought. Using an extracted implication $A \supset B$ gives rise to two subgoals: we have to find a normal proof of A, but we also still have to prove our overall goal, now with the additional extracted proposition B.

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\supset\mathrm{R} \qquad \frac{\Gamma, A \supset B \Longrightarrow A \qquad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\supset\mathrm{L}$$

Disjunction. This introduces no new considerations.

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\vee\mathrm{R}_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\vee\mathrm{R}_2$$

$$\frac{\Gamma, A \vee B, A \Longrightarrow C \qquad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\vee\mathrm{L}$$

Negation. Negation requires a judgment parametric in a proposition. Sometimes, this is encoded as an empty right-hand side (see Exercise **??**).

$$\frac{\Gamma, A \Longrightarrow p}{\Gamma \Longrightarrow \neg A}\neg\mathrm{R}^p \qquad \frac{\Gamma, \neg A \Longrightarrow A}{\Gamma, \neg A \Longrightarrow C}\neg\mathrm{L}$$

Truth. By our general method, there is no left rule, only a right rule which models the introduction rule.

$$\frac{}{\Gamma \Longrightarrow \top}\top\mathrm{R}$$

Falsehood. Again by our general method, there is no right rule, only a left rule which models the (upside-down) elimination rule.

$$\frac{}{\Gamma, \bot \Longrightarrow C}\ \bot\mathrm{L}$$

Universal Quantification. These require only a straightforward transcription, with the appropriate translation of the side condition.

$$\frac{\Gamma \Longrightarrow [a/x]A}{\Gamma \Longrightarrow \forall x.\ A}\ \forall\mathrm{R}^a \qquad \frac{\Gamma, \forall x.\ A, [t/x]A \Longrightarrow C}{\Gamma, \forall x.\ A \Longrightarrow C}\ \forall\mathrm{L}$$

Existential Quantification. Again, the rules can be directly constructed from the introduction and elimination rule of natural deduction.

$$\frac{\Gamma \Longrightarrow [t/x]A}{\Gamma \Longrightarrow \exists x.\ A}\ \exists\mathrm{R} \qquad \frac{\Gamma, \exists x.\ A, [a/x]A \Longrightarrow C}{\Gamma, \exists x.\ A \Longrightarrow C}\ \exists\mathrm{L}^a$$

The intended theorem describing the relationship between sequent calculus and natural deduction states that $\Gamma^{\downarrow} \vdash A \Uparrow$ if and only if $\Gamma \Longrightarrow A$. *Prima facie* is unlikely that we can prove either of these directions without further generalization, since the judgments $\Gamma^{\downarrow} \vdash A \Uparrow$ and $\Gamma^{\downarrow} \vdash A \downarrow$ are mutually recursive, and the statement above does not even mention the latter.

In preparation for the upcoming proof, we recall the general property of hypothetical judgments, namely that we can substitute a derivation of the appropriate judgment for a hypothesis. When applied to normal and extracting derivations, this yields the following property.

Lemma 3.5 (Substitution Property for Extractions)

1. *If* $\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash C \Uparrow$ *and* $\Gamma_1^{\downarrow} \vdash A \downarrow$ *then* $\Gamma_1^{\downarrow}, \Gamma_2^{\downarrow} \vdash C \Uparrow$.
2. *If* $\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash C \downarrow$ *and* $\Gamma_1^{\downarrow} \vdash A \downarrow$ *then* $\Gamma_1^{\downarrow}, \Gamma_2^{\downarrow} \vdash C \downarrow$.

Proof: By induction on the structure of the given derivations of $C \Uparrow$ and $C \downarrow$. In the case where the hypothesis is used we employ weakening, that is, we adjoin the additional hypotheses $\Gamma_2^{\downarrow}$ to every judgment in the derivation of $\Gamma_1^{\downarrow} \vdash A \downarrow$. □

Using this lemma, a direct proof goes through (somewhat surprisingly).

Theorem 3.6 (Soundness of Sequent Calculus)
If $\Gamma \Longrightarrow C$ *then* $\Gamma^{\downarrow} \vdash C \Uparrow$.

Proof: By induction on the structure of the given derivation $\mathcal{S}$. We show a few representative cases.

Case: Initial sequents.

$$\frac{}{\Gamma, C \Longrightarrow C}\,\text{init}$$

$\Gamma^{\downarrow}, u{:}C \downarrow \;\vdash C \downarrow$ By hypothesis u
$\Gamma^{\downarrow}, u{:}C \downarrow \;\vdash C \Uparrow$ By rule $\downarrow\Uparrow$

This case confirms that initial sequents correspond to the coercion from extractions to normal deductions.

Case: Implication right rule.

$$\frac{\begin{array}{c}\mathcal{S}_2\\ \Gamma, C_1 \Longrightarrow C_2\end{array}}{\Gamma \Longrightarrow C_1 \supset C_2}\,{\supset}\text{R}$$

$\Gamma^{\downarrow}, u{:}C_1 \downarrow \;\vdash C_2 \Uparrow$ By i.h. on $\mathcal{S}_2$
$\Gamma^{\downarrow} \vdash C_1 \supset C_2 \Uparrow$ By rule $\supset\text{I}^u$

This case exemplifies how right rules correspond directly to introduction rules.

Case: Implication left rule.

$$\frac{\begin{array}{c}\mathcal{S}_1\\ \Gamma, A_1 \supset A_2 \Longrightarrow A_1\end{array} \qquad \begin{array}{c}\mathcal{S}_2\\ \Gamma, A_1 \supset A_2, A_2 \Longrightarrow C\end{array}}{\Gamma, A_1 \supset A_2 \Longrightarrow C}\,{\supset}\text{L}$$

$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow \;\vdash A_1 \Uparrow$ By i.h. on $\mathcal{S}_1$
$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow \;\vdash A_1 \supset A_2 \downarrow$ By hypothesis u
$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow \;\vdash A_2 \downarrow$ By rule $\supset$E
$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow, w{:}A_2 \downarrow \;\vdash C \Uparrow$ By i.h. on $\mathcal{S}_2$
$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow \;\vdash C \Uparrow$ By substitution property (Lemma 3.5)

This case illustrates how left rules correspond to elimination rules. The general pattern is that the result of applying the appropriate elimination rule is substituted for a hypothesis.

□

The proof of completeness is somewhat trickier—we first need to generalize the induction hypothesis. Generalizing a desired theorem so that a direct inductive proof is possible often requires considerable ingenuity and insight into the problem. In this particular case, the generalization is of medium difficulty. The reader who has not seen the proof is invited to test his understanding by carrying out the generalization and proof himself before reading on.

The nature of a sequent as a hypothetical judgment gives rise to several general properties we will take advantage of. We make two of them, weakening and contraction, explicit in the following lemma.

Lemma 3.7 (Structural Properties of Sequents)

1. *(Weakening) If* $\Gamma \Longrightarrow C$ *then* $\Gamma, A \Longrightarrow C$.
2. *(Contraction) If* $\Gamma, A, A \Longrightarrow C$ *then* $\Gamma, A \Longrightarrow C$.

Proof: First, recall our general convention that we consider the hypotheses of a sequent modulo permutation. We prove each property by a straightforward induction over the structure of the derivation. In the case of weakening we adjoin an unused hypothesis A *left* to each sequent in the derivation. In the case of contraction we replace any use of either of the two hypotheses by a common hypothesis. □

The theorem below only establishes the completeness of sequent derivations with respect to normal deductions. That is, at this point we have not established the completeness of sequents with respect to arbitrary natural deductions which is more difficult.

Theorem 3.8 (Completeness of Sequent Derivations)

1. *If* $\Gamma^{\downarrow} \vdash C \Uparrow$ *then* $\Gamma \Longrightarrow C$.
2. *If* $\Gamma^{\downarrow} \vdash A \downarrow$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$.

Proof: By induction on the structure of the given derivations $\mathcal{I}$ and $\mathcal{E}$. We show some representative cases.

Case: Use of hypotheses.

$$\mathcal{E} = \frac{}{\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash A \downarrow}\, u$$

$\Gamma_1, A, \Gamma_2, A \Longrightarrow C$	Assumption
$\Gamma_1, A, \Gamma_2 \Longrightarrow C$	By contraction (Lemma 3.7)

Case: Coercion.

$$\mathcal{I} = \frac{\begin{array}{c}\mathcal{E}\\ \Gamma^{\downarrow} \vdash C \downarrow\end{array}}{\Gamma^{\downarrow} \vdash C \Uparrow}\, {\downarrow}{\Uparrow}$$

$\Gamma, C \Longrightarrow C$	By rule init
$\Gamma \Longrightarrow C$	By i.h. on $\mathcal{E}$

Case: Implication introduction.

$$\mathcal{I} = \frac{\begin{array}{c}\mathcal{I}_2\\ \Gamma^{\downarrow}, u{:}C_1 \downarrow \vdash C_2 \Uparrow\end{array}}{\Gamma^{\downarrow} \vdash C_1 \supset C_2 \Uparrow}\, {\supset}\mathrm{I}^u$$

$\Gamma, C_1 \Longrightarrow C_2$	By i.h. on $\mathcal{I}_2$
$\Gamma \Longrightarrow C_1 \supset C_2$	By rule $\supset$R

Case: Implication elimination.

$$\mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_2\\ \Gamma^{\downarrow} \vdash A_1 \supset A_2 \downarrow\end{array} \qquad \begin{array}{c}\mathcal{I}_1\\ \Gamma^{\downarrow} \vdash A_1 \Uparrow\end{array}}{\Gamma^{\downarrow} \vdash A_2 \downarrow} \supset\text{E}$$

$\Gamma, A_2 \Longrightarrow C$	Assumption
$\Gamma, A_1 \supset A_2, A_2 \Longrightarrow C$	By weakening (Lemma 3.7)
$\Gamma \Longrightarrow A_1$	By i.h. on $\mathcal{I}_1$
$\Gamma, A_1 \supset A_2 \Longrightarrow A_1$	By weakening (Lemma 3.7)
$\Gamma, A_1 \supset A_2 \Longrightarrow C$	By rule $\supset$L
$\Gamma \Longrightarrow C$	By i.h. on $\mathcal{E}_2$

□

In order to establish soundness and completeness with respect to arbitrary natural deductions we establish a connection to annotated natural deductions. Recall that this is an extension of normal deductions which we showed sound and complete with respect to arbitrary natural deduction in Theorems 3.2 and 3.3. We related annotated natural deductions to the sequent calculus by adding a rule called cut.

We write the extended judgment of sequent derivations with cut as $\Gamma \overset{+}{\Longrightarrow} C$. It is defined by copies of all the rules for $\Gamma \Longrightarrow C$, plus the rule of cut:

$$\dfrac{\Gamma \overset{+}{\Longrightarrow} A \qquad \Gamma, A \overset{+}{\Longrightarrow} C}{\Gamma \overset{+}{\Longrightarrow} C}\,\text{cut}$$

Thought of from the perspective of bottom-up proof construction, this rule corresponds to proving and then assuming a lemma A during a derivation.

Theorem 3.9 (Soundness of Sequent Calculus with Cut)
If $\Gamma \overset{+}{\Longrightarrow} C$ *then* $\Gamma^{\downarrow} \vdash^{+} C \Uparrow$.

Proof: As in Theorem 3.6 by induction on the structure of the given derivation $\mathcal{S}$, with one additional case.

Case: Cut.

$$\mathcal{S} = \dfrac{\begin{array}{c}\mathcal{S}_1\\ \Gamma \Longrightarrow A\end{array} \qquad \begin{array}{c}\mathcal{S}_2\\ \Gamma, A \Longrightarrow C\end{array}}{\Gamma \Longrightarrow C}\,\text{cut}$$

$\Gamma^{\downarrow} \vdash^{+} A \Uparrow$ By i.h. on $\mathcal{S}_1$
$\Gamma^{\downarrow} \vdash^{+} A \downarrow$ By rule $\Uparrow\downarrow$
$\Gamma^{\downarrow}, u{:}A \downarrow \vdash^{+} C \Uparrow$ By i.h. on $\mathcal{S}_2$
$\Gamma^{\downarrow} \vdash^{+} C \Uparrow$ By substitution (Lemma 3.5, generalized)

We see that, indeed, cut corresponds to the coercion from normal to extraction derivations.

□

Theorem 3.10 (Completeness of Sequent Calculus with Cut)

1. *If* $\Gamma^{\downarrow} \vdash^{+} C \Uparrow$ *then* $\Gamma \stackrel{+}{\Longrightarrow} C$.
2. *If* $\Gamma^{\downarrow} \vdash^{+} A \downarrow$ *and* $\Gamma, A \stackrel{+}{\Longrightarrow} C$ *then* $\Gamma \stackrel{+}{\Longrightarrow} C$.

Proof: As in the proof of Theorem 3.10 with one additional case.

Case: Coercion from normal to extraction derivations.

$$\mathcal{E} = \cfrac{\begin{array}{c}\mathcal{I}\\ \Gamma^{\downarrow} \vdash^{+} A \Uparrow\end{array}}{\Gamma^{\downarrow} \vdash^{+} A \downarrow}\Uparrow\downarrow$$

$\Gamma \Longrightarrow A$ By i.h. on $\mathcal{I}$
$\Gamma, A \Longrightarrow C$ By assumption
$\Gamma \Longrightarrow C$ By rule cut

□

The central property of the sequent calculus is that the cut rule is redundant. That is, if $\Gamma \stackrel{+}{\Longrightarrow} C$ then $\Gamma \Longrightarrow C$. This so-called cut elimination theorem (Gentzen's *Hauptsatz* [Gen35]) is one of the central theorems of logic. As an immediately consequence we can see that not every proposition has a proof, since no rule is applicable to derive $\cdot \Longrightarrow \bot$. In the system with cut, a derivation of this sequent might end in the cut rule and consistency is not at all obvious. The proof of cut elimination and some of its many consequences are the subject of the next section.

3.4 Exercises

Exercise 3.1 Consider a system of normal deduction where the elimination rules for disjunction and existential are allowed to end in an extraction judgment.

$$\frac{\Gamma^{\downarrow} \vdash A \vee B \downarrow \qquad \Gamma^{\downarrow}, u{:}A \downarrow\ \vdash C \downarrow \qquad \Gamma^{\downarrow}, w{:}B \downarrow\ \vdash C \downarrow}{\Gamma^{\downarrow} \vdash C \downarrow} \vee\mathrm{E}^{u,w}$$

$$\frac{\Gamma^{\downarrow} \vdash \exists x.\ A \downarrow \qquad \Gamma^{\downarrow}, u{:}[a/x]A \downarrow\ \vdash C \downarrow}{\Gamma^{\downarrow} \vdash C \downarrow} \exists\mathrm{E}^{a,u}$$

Discuss the relative merits of allowing or disallowing these rules and show how they impact the subsequent development in this Chapter (in particular, bi-directional type-checking and the relationship to the sequent calculus).

Exercise 3.2

1. Give an example of a natural deduction which is *not* normal (in the sense defined in Section 3.1), yet contains no subderivation which can be locally reduced.
2. Generalizing from the example, devise additional rules of reduction so that any natural deduction which is not normal can be reduced. You should introduce no more and no fewer rules than you need for this purpose.
3. Prove that your rules satisfy the specification in part (2).

Exercise 3.3 Write out the rules defining the judgments $\Gamma^{\downarrow} \vdash^{+} I : A \Uparrow$ and $\Gamma^{\downarrow} \vdash^{+} E : A \downarrow$ and prove Theorem 3.4. Make sure to carefully state the induction hypothesis (if it is different from the statement of the theorem) and consider all the cases.