Chapter 4

Focused Derivations

The sequent calculus as presented in the previous chapter is an excellent foundation for proof search strategies, but it is not yet practical. For a typical sequent there are many choices, such as which left or right rule to use to reduce the goal in the bottom-up construction of a proof. After one step, similar choices arise again, and so on. Without techniques to eliminate some of this non-determinism one would be quickly overwhelmed with multiple choices.

In this chapter we present two techniques to reduce the amount of non-determinism in search. The first are *inversion properties* which hold when the premises of an inference rule are derivable if and only if the conclusion is. This means that we do not lose completeness when applying an invertible rule as soon as it is applicable. The second are *focusing properties* which allow us to chain together non-invertible inference rules with consecutive principal formulas, once again without losing completeness.

While inversion and focusing are motivated by bottom-up proof search, they generally reduce the number of derivations in the search space. For this reason they also apply in top-down search procedures such as the inverse method introduced in Chapter **??**.

4.1 Inversion

The simplest way to avoid non-determinism is to consider those propositions on the left or right for which there is a unique way to apply a corresponding left or right rule. For example, to prove $A \wedge B$ we can immediately apply the right rule without losing completeness. On the other hand, to prove $A \vee B$ we can not immediately apply a left rule. As a counterexample consider $B \vee A \Longrightarrow A \vee B$, where we first need to apply a left rule.

On a given sequent, a number of invertible rules may be applicable. However, the order of this choice does not matter. In other words, we have replaced *don't-know* non-determinism by *don't-care* non-determinism.

Determining the invertibility of left rules in order to support this strategy requires some additional considerations. The pure inversion property states that the premises should be derivable if and only if the conclusion is. However, in left rule the principal formula is still present in the premises, which means we can continue to apply the same left rule over and over again leading to non-termination. So we require in addition that the principal formula of a left rule is no longer needed, thereby guaranteeing the termination of the inversion phase of the search.

Theorem 4.1 (Inversion)

1. *If* $\Gamma \Longrightarrow A \wedge B$ *then* $\Gamma \Longrightarrow A$ *and* $\Gamma \Longrightarrow B$.
2. *If* $\Gamma \Longrightarrow A \supset B$ *then* $\Gamma, A \Longrightarrow B$.
3. *If* $\Gamma \Longrightarrow \forall x.\, A$ *then* $\Gamma \Longrightarrow [a/x]A$ *for a new individual parameter* a.
4. *If* $\Gamma \Longrightarrow \neg A$ *then* $\Gamma, A \Longrightarrow p$ *for a new propositional parameter* p.
5. *If* $\Gamma, A \wedge B \Longrightarrow C$ *then* $\Gamma, A, B \Longrightarrow C$.
6. *If* $\Gamma, \top \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$.
7. *If* $\Gamma, A \vee B \Longrightarrow C$ *then* $\Gamma, A \Longrightarrow C$ *and* $\Gamma, B \Longrightarrow C$.
8. *If* $\Gamma, \exists x.\, A \Longrightarrow C$ *then* $\Gamma, [a/x]A \Longrightarrow C$ *for a new individual parameter* a.

Proof: By induction over the structure of the given derivations. Parts (5) and (6) are somewhat different in that they extract and inversion property from two and zero left rules, respectively. The proof is nonetheless routine. □

The rules $\top$R and $\perp$L are a special case: they can be applied eagerly without losing completeness, but these rules have no premises and therefore do not admit a theorem of the form above. None of the other rules permit an inversion property, as the following counterexamples show. These counterexamples can easily be modifed so that they are not initial sequents.

1. $A \vee B \Longrightarrow A \vee B$ (both $\vee\mathrm{R}_1$ or $\vee\mathrm{R}_2$ lead to an unprovable sequent).
2. $\perp \Longrightarrow \perp$ (no right rule applicable).
3. $\exists x.\, A \Longrightarrow \exists x.\, A$ ($\exists$R leads to an unprovable sequent).
4. $A \supset B \Longrightarrow A \supset B$ ($\supset$L leads to an unprovable sequent).
5. $\neg A \Longrightarrow \neg A$ ($\neg$L leads to an unprovable sequent).
6. $\forall x.\, A \Longrightarrow \forall x.\, A$ ($\forall$L leads to an unprovable sequent if we erase the original copy of $\forall x.\, A$).

Now we can write out a pure inversion strategy in the form of an inference system. One difficulty with such a system is that the don't-care non-determinism is not directly visible and has to be remarked on separately. We also refer to don't-care non-determinism as *conjunctive non-determinism*: eventually, all applicable rules have to be applied, but their order is irrelevant as far as provability is concerned.

First, we distinguish those kinds of propositions for which either the left or the right rule is *not* invertible. We call them *passive* propositions (either on the left or on the right).[1]

$$
\begin{array}{rccl}
\text{Left passive propositions} & L & ::= & P \mid A_1 \supset A_2 \mid \forall x.\, A \\
\text{Right passive propositions} & R & ::= & P \mid A_1 \vee A_2 \mid \bot \mid \exists x.\, A \\
\text{Passive antecedents} & \Delta & ::= & \cdot \mid \Delta, L
\end{array}
$$

We also write L^+ and R^+ for non-atomic left and right passive propositions, respectively. Sequents are then composed of four judgments: left and right propositions, each of which may be active or passive. In order to simplify the notation, we collect like judgments into zones, keeping in mind that there can only be one proposition on the right. Sequents are then written as

$$\Delta; \Gamma \Longrightarrow A; \cdot \quad \text{or} \quad \Delta; \Gamma \Longrightarrow \cdot; R$$

where the outer zones containing Δ or R are passive and the inner zones containing Γ or A are active. We still think of Δ and Γ as unordered and omit labels for the sake of brevity. We break down the principal connectives in the active propositions eagerly until we have reduced the sequent to one with only passive propositions. At that point we have to choose a left rule to apply and then iterate the process.

Right Active Propositions.

$$
\frac{\Delta; \Gamma \Longrightarrow A; \cdot \qquad \Delta; \Gamma \Longrightarrow B; \cdot}{\Delta; \Gamma \Longrightarrow A \wedge B; \cdot}\wedge\mathrm{R}
\qquad
\frac{}{\Delta; \Gamma \Longrightarrow \top}\top\mathrm{R}
$$

$$
\frac{\Delta; \Gamma, A \Longrightarrow B; \cdot}{\Delta; \Gamma \Longrightarrow A \supset B; \cdot}\supset\mathrm{R}
\qquad
\frac{\Delta; \Gamma \Longrightarrow [a/x]A; \cdot}{\Delta; \Gamma \Longrightarrow \forall x.\, A; \cdot}\forall\mathrm{R}^a
$$

Left Active Propositions. In order to avoid duplicating the rules depending on a passive or active succedent, we write ρ to stand for either $A; \cdot$ or $\cdot; R$.

[1] *[for the moment, we do not consider negation explicitly, but think of it as defined]*

$$\frac{\Delta; \Gamma, A, B \Longrightarrow \rho}{\Delta; \Gamma, A \wedge B \Longrightarrow \rho}\,\wedge\mathrm{L} \qquad \frac{\Delta; \Gamma \Longrightarrow \rho}{\Delta; \Gamma, \top \Longrightarrow \rho}\,\top\mathrm{L}$$

$$\frac{\Delta; \Gamma, A \Longrightarrow \rho \qquad \Delta; \Gamma, B \Longrightarrow \rho}{\Delta; \Gamma, A \vee B \Longrightarrow \rho}\,\vee\mathrm{L} \qquad \frac{}{\Delta; \Gamma, \top \Longrightarrow \rho}\,\bot\mathrm{L}$$

$$\frac{\Delta; \Gamma, [a/x]A \Longrightarrow \rho}{\Delta; \Gamma, \exists x.\, A \Longrightarrow \rho}\,\exists\mathrm{L}^a$$

Transitions. These rules can be applied to move passive propositions which have been uncovered to their appropriate zones.

$$\frac{\Delta; \Gamma \Longrightarrow \cdot; R}{\Delta; \Gamma \Longrightarrow R; \cdot}\,\mathrm{tR} \qquad \frac{(\Delta, L); \Gamma \Longrightarrow \rho}{\Delta; (\Gamma, L) \Longrightarrow \rho}\,\mathrm{tL}$$

Right Passive Propositions. The active and transition rules always terminate when applied in a bottom-up fashion during proof search (see Lemma 4.3). Moverover, they can be applied in any order until sequents of the form $\Delta; \cdot \Longrightarrow \cdot; R$ are reached. Now a don't-know non-deterministic choice arises: either we apply a right rule to infer R or a left rule to one of the passive assumptions in Δ. We also refer to don't-know non-determinism as *disjunctive non-determinism* since we have to pick one of several possibilities, but one is sufficient.

$$\frac{\Delta; \cdot \Longrightarrow A; \cdot}{\Delta; \cdot \Longrightarrow \cdot; A \vee B}\,\vee\mathrm{R}_1 \qquad \frac{\Delta; \cdot \Longrightarrow B; \cdot}{\Delta; \cdot \Longrightarrow \cdot; A \vee B}\,\vee\mathrm{R}_2$$

$$\textit{no rule for } \bot \qquad \frac{\Delta; \cdot \Longrightarrow [t/x]A; \cdot}{\Delta; \cdot \Longrightarrow \cdot; \exists x.\, A}\,\exists\mathrm{R}$$

Left Passive Propositions. Left passive propositions may be needed more than once, so they are duplicated in the application of the left rules.[2]

$$\frac{\Delta, A \supset B; \cdot \Longrightarrow A; \cdot \qquad \Delta, A \supset B; B \Longrightarrow \cdot; R}{\Delta, A \supset B; \cdot \Longrightarrow \cdot; R}\,\supset\mathrm{L}$$

$$\frac{\Delta, \forall x.\, A; [t/x]A \Longrightarrow \cdot; R}{\Delta, \forall x.\, A; \cdot \Longrightarrow \cdot; R}\,\forall\mathrm{L}$$

[2] [*some optimization may be possible here.*]

Initial Sequents. This leaves the question of initial sequents, which is easily handled by allowing an left passive atomic proposition to match a right passive atomic proposition.

$$\frac{}{\Delta, P; \cdot \Longrightarrow \cdot; P}\,\text{init}$$

The judgments $\Delta; \Gamma \Longrightarrow \rho$ are hypothetical in Δ, but *not* hypothetical in Γ. This is because proposition in Γ do not persist, and because they have to be empty in the initial sequents. In other words, contraction and weakening are not available for Γ. However, it can be explained as a *linear hypothetical judgment* where each linear hypothesis must be used exactly once in a derivation. We do not formalize this notion any further, but just remark that appropriate versions of the substitution property can be devised to explain its meaning.

First, the soundness theorem is straightforward, since inversion proofs merely eliminate some disjunctive non-determinism.

Theorem 4.2 (Soundness of Inversion Proofs)
If $\Delta; \Gamma \Longrightarrow A; \cdot$ *or* $\Delta; \Gamma \Longrightarrow \cdot; A$ *then* $\Delta, \Gamma \Longrightarrow A$.

Proof: By a straightforward induction over the given derivation, applying weakening in some cases. □

Formulating appropriate theorems for the study of inversion proofs is somewhat difficult, because of the nature conjunctive and disjunctive non-determinism. If we think of an unproven sequent as a *goal* and the unproven leaves of a partially constructed derivation as *subgoals.*

Lemma 4.3 (Termination of Inversion)
Given a goal $\Delta; \Gamma \Longrightarrow A; \cdot$ *or* $\Delta; \Gamma \Longrightarrow \cdot; A$. *Any sequence of applications of right active, left active, or transition rules terminates. Moreover, for each subgoal* $\Delta'; \Gamma' \Longrightarrow A'; \cdot$ *and* $\Delta'; \Gamma' \Longrightarrow \cdot; A'$*, the sequent* $\Delta', \Gamma' \Longrightarrow A'$ *is derivable if and only if* $\Delta, \Gamma \Longrightarrow A$ *is derivable.*

Proof: Termination follows by induction on the sum of the number of logical connectives and quantifiers in Γ and A. Preservation of provability follows by the inversion properties (Theorem 4.1). □

The first completeness theorem below does not express the conjunctive non-determinism in the search for inversion proofs.

Theorem 4.4 (Completeness of Inversion Proofs)
If $\Gamma \Longrightarrow A$ *then* $\cdot; \Gamma \Longrightarrow A; \cdot$.

Proof: [*to be filled in*]

□