Chapter 3

Sequent Calculus

In this chapter we develop the sequent calculus as a formal system for proof search in natural deduction. The sequent calculus was originally introduced by Gentzen [Gen35], primarily as a technical device for proving consistency of predicate logic. Our goal of describing a proof search procedure for natural deduction predisposes us to a formulation due to Kleene [Kle52] called G_3.

We introduce the sequent calculus in two steps. The first step is based on the simple strategy of building a natural deduction by using introduction rules bottom-up and elimination rules top-down. The result is an intercalation calculus [**?**]. The second step consists of reformulating the rules for intercalation so that both forms of rules work bottom-up, resulting in the sequent calculus.

We also show how intercalation derivations lead to more compact proof terms, and how to extract proof terms from sequent calculus derivations.

3.1 Intercalation

A simple strategy in the search for a natural deduction is to use introduction rules reasoning bottom-up (from the proposed theorem towards the hypotheses) and the elimination rules top-down (from the assumptions towards the proposed theorem). When they meet in the middle we have found a *normal* deduction. Towards the end of this chapter we show that this strategy is in fact complete: if a proposition A has a natural deduction then it has a normal deduction. First, however, we need to make this strategy precise.

A general technique for representing proof search strategies is to introduce new judgments which permit only those derivations which can be found by the intended strategy. We then prove the correctness of the new, restricted judgments by appropriate soundness and completeness theorems.

In this case, we introduce two judgments:

$A \Uparrow$ Proposition A has a normal deduction, and

$A \downarrow$ Proposition A is extracted from a hypothesis.

They are defined by restricting the rules of natural deduction according to their status as introduction or elimination rules. Hypotheses can be trivially extracted. Therefore the necessary hypothetical judgments (in localized form, see Section 2.3) are

$u_1{:}A_1 \downarrow, \ldots, u_n{:}A_n \downarrow \;\vdash A \Uparrow$ and

$u_1{:}A_1 \downarrow, \ldots, u_n{:}A_n \downarrow \;\vdash A \downarrow$.

We write $\Gamma^{\downarrow}$ for a context of the form shown above.

Hypotheses. The general rule for hypotheses simply reflects the nature of hypothetical judgments.

$$\frac{}{\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash A \downarrow}\, u$$

Coercion. The bottom-up and top-down derivations must be able to meet in the middle.

$$\frac{\Gamma^{\downarrow} \vdash A \downarrow}{\Gamma^{\downarrow} \vdash A \Uparrow}\, {\downarrow}{\Uparrow}$$

Looked at another way, this rule allows us to coerce any extraction derivation to a normal deduction. Of course, the opposite coercion would contradict the intended strategy.

Conjunction. The rules for conjunction exhibit no unexpected features: the introduction rule is classified as a bottom-up rule, the elimination rule is classified as a top-down rule.

$$\frac{\Gamma^{\downarrow} \vdash A \Uparrow \qquad \Gamma^{\downarrow} \vdash B \Uparrow}{\Gamma^{\downarrow} \vdash A \wedge B \Uparrow}\, \wedge\mathrm{I}$$

$$\frac{\Gamma^{\downarrow} \vdash A \wedge B \downarrow}{\Gamma^{\downarrow} \vdash A \downarrow}\, \wedge\mathrm{E_L} \qquad \frac{\Gamma^{\downarrow} \vdash A \wedge B \downarrow}{\Gamma^{\downarrow} \vdash B \downarrow}\, \wedge\mathrm{E_R}$$

Truth. For truth, there is only an introduction rule which is classified as normal.

$$\frac{}{\Gamma^{\downarrow} \vdash \top \Uparrow}\, \top\mathrm{I}$$

Implication. The introduction rule for implication is straightforward. In the elimination rule we require that the the second premise is normal. It is only the first premise (whose primary connective is eliminated in this rule) which must be extracted from a hypothesis.

$$\frac{\Gamma^{\downarrow}, u{:}A \downarrow \;\vdash B \Uparrow}{\Gamma^{\downarrow} \vdash A \supset B \Uparrow}\, {\supset}\mathrm{I}^u \qquad \frac{\Gamma^{\downarrow} \vdash A \supset B \downarrow \qquad \Gamma^{\downarrow} \vdash A \Uparrow}{\Gamma^{\downarrow} \vdash B \downarrow}\, {\supset}\mathrm{E}$$

Disjunction. The introduction rules for disjunction are straightforward. For the elimination rule, again the premise with the connective which is eliminated must have a top-down derivation. The new assumptions in each branch also are top-down derivations. Overall, for the derivation to be normal we must require the derivations of both premises to be normal.

$$\frac{\Gamma^{\downarrow} \vdash A \Uparrow}{\Gamma^{\downarrow} \vdash A \vee B \Uparrow} \vee\mathrm{I_L} \qquad \frac{\Gamma^{\downarrow} \vdash B \Uparrow}{\Gamma^{\downarrow} \vdash A \vee B \Uparrow} \vee\mathrm{I_R}$$

$$\frac{\Gamma^{\downarrow} \vdash A \vee B \downarrow \qquad \Gamma^{\downarrow}, u{:}A \downarrow\ \vdash C \Uparrow \qquad \Gamma^{\downarrow}, w{:}B \downarrow\ \vdash C \Uparrow}{\Gamma^{\downarrow} \vdash C \Uparrow} \vee\mathrm{E}^{u,w}$$

It would also consistent to allow the derivations of C to be extractions, but it is not necessary to obtain a complete search procedure and complicates the relation to the sequent calculus (see Exercise 3.1).

Falsehood. Falsehood corresponds to a disjunction with no alternatives. Therefore there is no introduction rule, and the elimination rule has no cases. This consideration yields

$$\frac{\Gamma^{\downarrow} \vdash \bot \downarrow}{\Gamma^{\downarrow} \vdash C \Uparrow} \bot\mathrm{E}.$$

For this rule, it does not appear to make sense to allow the conclusion as having been constructed top-down, since the proposition C would be completely unrestricted.

Negation. Negation combines elements from implication and falsehood, since we may think of $\neg A$ as $A \supset \bot$.[1]

$$\frac{\Gamma^{\downarrow}, u{:}A \downarrow\ \vdash p \Uparrow}{\Gamma^{\downarrow} \vdash \neg A \Uparrow} \neg\mathrm{I}^{p,u} \qquad \frac{\Gamma^{\downarrow} \vdash \neg A \downarrow \qquad \Gamma^{\downarrow} \vdash A \Uparrow}{\Gamma^{\downarrow} \vdash C \Uparrow} \neg\mathrm{E}$$

Universal Quantification. Universal quantification does not introduce any new considerations.

$$\frac{\Gamma^{\downarrow} \vdash [a/x]A \Uparrow}{\Gamma^{\downarrow} \vdash \forall x.\ A \Uparrow} \forall\mathrm{I}^{a} \qquad \frac{\Gamma^{\downarrow} \vdash \forall x.\ A \downarrow}{\Gamma^{\downarrow} \vdash [t/x]A \downarrow} \forall\mathrm{E}$$

Existential Quantification. Existential quantification is similar to disjunction and a more lenient view of extraction is possible here, too (see Exercise 3.1).

$$\frac{\Gamma^{\downarrow} \vdash [t/x]A \Uparrow}{\Gamma^{\downarrow} \vdash \exists x.\ A \Uparrow} \exists\mathrm{I} \qquad \frac{\Gamma^{\downarrow} \vdash \exists x.\ A \downarrow \qquad \Gamma^{\downarrow}, u{:}[a/x]A \downarrow\ \vdash C \Uparrow}{\Gamma^{\downarrow} \vdash C \Uparrow} \exists\mathrm{E}^{a,u}$$

[1] *[reconsider]*

It is quite easy to see that normal and extraction derivations are sound with respect to natural deduction. In order to state and prove this theorem, we introduce some conventions. Given a context

$$\Gamma^{\downarrow} = u_1{:}A_1 \downarrow, \ldots, u_n{:}A_n \downarrow$$

we denote

$$u_1{:}A_1, \ldots, u_n{:}A_n$$

by Γ and vice versa.

Theorem 3.1 (Soundness of Normal Deductions)

1. *If* $\Gamma^{\downarrow} \vdash A \Uparrow$ *then* $\Gamma \vdash A$, *and*
2. *if* $\Gamma^{\downarrow} \vdash A \downarrow$ *then* $\Gamma \vdash A$.

Proof: By induction on the structure of the given derivations. We show only three cases, since the proof is absolutely straightforward.

Case:

$$\mathcal{E} = \frac{}{\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash A \downarrow} u$$

The we construct directly $\Gamma_1, u{:}A, \Gamma_2 \vdash A$.

Case:

$$\mathcal{N} = \frac{\begin{array}{c}\mathcal{E}\\ \Gamma^{\downarrow} \vdash A \downarrow\end{array}}{\Gamma^{\downarrow} \vdash A \Uparrow} \downarrow\Uparrow$$

Then $\Gamma \vdash A$ by induction hypothesis on $\mathcal{E}$.

Case:

$$\mathcal{N} = \frac{\begin{array}{c}\mathcal{N}_2\\ \Gamma^{\downarrow}, u{:}A_1 \downarrow \vdash A_2 \Uparrow\end{array}}{\Gamma^{\downarrow} \vdash A_1 \supset A_2 \Uparrow} \supset\mathrm{I}^u$$

$\Gamma, u{:}A_1 \vdash A_2$	By i.h. on $\mathcal{N}_2$
$\Gamma \vdash A_1 \supset A_2$	By rule $\supset$I

□

When trying to give a translation in the other direction we encounter a difficulty: certain patterns of inference cannot be annotated directly. For example, consider

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B} \wedge\mathrm{I}}{\Gamma \vdash A} \wedge\mathrm{E_L}.$$

If we try to classify each judgment, we obtain a conflict:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Gamma \vdash A \Uparrow\end{array} \qquad \begin{array}{c}\mathcal{E}'\\ \Gamma \vdash B \Uparrow\end{array}}{\Gamma \vdash A \wedge B\ ?}\wedge\mathrm{I}}{\Gamma \vdash A \downarrow}\wedge\mathrm{E_L}.$$

In this particular case, we can avoid the conflict: in order to obtain the derivation of $A \Uparrow$ we can just translate the derivation $\mathcal{D}$ and avoid the final two inferences! In general, we can try to apply local reductions to the given original derivation until no situations of the form above remain. This approach is called *normalization*. It is not easy to prove that normalization terminates, and the situation is complicated by the fact that the local reductions alone do not suffice to transform an arbitrary natural deduction into normal form (see Exercise 3.2).

Here, we follow an alternative approach to prove completeness of normal deductions. First, we temporarily augment the system with another rule which makes the translation from natural deductions immediate. Then we relate the resulting system to a sequent calculus and show that the additional rule was redundant.

A candidate for the additional rule is easy to spot: we just add the missing coercion from normal to extraction deductions. Since all rules are present, we can just coerce back and forth as necessary in order to obtain a counterpart for any natural deduction in this extended system. Of course, the resulting derivations are no longer normal, which we indicate by decorating the turnstile with a "+". The judgments $\Gamma^{\downarrow} \vdash^{+} A \Uparrow$ and $\Gamma^{\downarrow} \vdash^{+} A \downarrow$ are defined by all counterparts of all rules which define normal and extracting derivations, plus the rule

$$\dfrac{\Gamma^{\downarrow} \vdash^{+} A \Uparrow}{\Gamma^{\downarrow} \vdash^{+} A \downarrow}\Uparrow\downarrow$$

Now the annotation in the example above can be completed.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Gamma \vdash^{+} A \Uparrow\end{array} \qquad \begin{array}{c}\mathcal{E}'\\ \Gamma \vdash^{+} B \Uparrow\end{array}}{\Gamma \vdash^{+} A \wedge B \Uparrow}\wedge\mathrm{I}}{\Gamma \vdash^{+} A \wedge B \downarrow}\Uparrow\downarrow}{\Gamma \vdash^{+} A \downarrow}\wedge\mathrm{E_L}$$

Both soundness and completeness of the extended calculus with respect to natural deduction is easy to see.

Theorem 3.2 (Soundness of Annotated Deductions)

1. *If* $\Gamma^{\downarrow} \vdash^{+} A \Uparrow$ *then* $\Gamma \vdash A$*, and*

2. *if* $\Gamma^{\downarrow} \vdash^{+} A \downarrow$ *then* $\Gamma \vdash A$.

Proof: By simultaneous induction over the structure of the given derivations. □

The constructive proof of the completeness theorem below will contain an algorithm for annotating a given natural deduction.

Theorem 3.3 (Completeness of Annotated Deductions)

1. *If* $\Gamma \vdash A$ *then* $\Gamma^{\downarrow} \vdash^{+} A \Uparrow$, *and*

2. *if* $\Gamma \vdash A$ *then* $\Gamma^{\downarrow} \vdash^{+} A \downarrow$.

Proof: By induction over the structure of the given derivation. We show only two cases.

Case:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash B \supset A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A} \supset\mathrm{E}$$

$\Gamma^{\downarrow} \vdash^{+} B \supset A \downarrow$	By i.h. (2) on $\mathcal{D}$
$\Gamma^{\downarrow} \vdash^{+} B \Uparrow$	By i.h. (1) on $\mathcal{E}$
$\Gamma^{\downarrow} \vdash^{+} A \downarrow$	By rule $\supset$E, proving (2)
$\Gamma^{\downarrow} \vdash^{+} A \Uparrow$	By rule $\downarrow\Uparrow$, proving (1)

Case:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma, u{:}A_1 \vdash A_2\end{array}}{\Gamma \vdash A_1 \supset A_2} \supset\mathrm{I}^u$$

$\Gamma^{\downarrow}, u{:}A_1 \downarrow\ \vdash^{+} A_2 \Uparrow$	By i.h. (1) on $\mathcal{D}_2$
$\Gamma^{\downarrow} \vdash^{+} A_1 \supset A_2 \Uparrow$	By rule $\supset\mathrm{I}^u$, proving (1)
$\Gamma^{\downarrow} \vdash^{+} A_1 \supset A_2 \downarrow$	By rule $\Uparrow\downarrow$, proving (2)

□

Even though natural deductions and annotated deductions are very similar, they are not in bijective correspondence. For example, in an annotated deduction we can simply alternate the two coercions an arbitrary number of times. Under the translation to natural deduction, all of these are identified.

Before we introduce the sequent calculus, we make a brief excursion to study the impact of annotations on proof terms.

3.2 Compact Proof Terms

The proof terms introduced in Section 2.4 sometimes contain significant amounts of redundant information. The reason are the propositions which label λ-abstractions and also occur in the inl^A, inr^A, $\mu^p u{:}A$, $\cdot_A$, and abort^A constructs. For example, assume we are given a proof term $\lambda u{:}A.\, M$ and we are supposed to check if it represents a proof of $A' \supset B$. We then have to check that $A = A'$ and, moreover, the information is duplicated. The reason for this duplication was the intended invariant that every term proves a unique proposition. Under the interpretations of propositions as types, this means we can always synthesize a unique type for every valid term. However, we can improve this if we alternate between synthesizing a type and checking a term against a given type.

Therefore we introduce two classes of terms: those whose type can be synthesized, and those which can be checked against a type. Interestingly, this corresponds precisely with the annotations as introduction or elimination rules given above. We ignore negation again, thinking of $\neg A$ as $A \supset \bot$. We already discussed why the eliminations for disjunction and falsehood appear among the intro terms.

$$
\begin{array}{llll}
\text{Intro Terms} & I & ::= & \langle I_1, I_2\rangle & \text{Conjunction}\\
& & & \mid \lambda u.\, I & \text{Implication}\\
& & & \mid \mathrm{inl}\, I \mid \mathrm{inr}\, I & \text{Disjunction}\\
& & & \mid (\,\mathbf{case}\ E\ \mathbf{of}\ \mathrm{inl}\, u_1 \Rightarrow I_1 \mid \mathrm{inr}\, u_2 \Rightarrow I_2) & \\
& & & \mid \langle\,\rangle & \text{Truth}\\
& & & \mid \mathrm{abort}\, E & \text{Falsehood}\\
& & & \mid E & \text{Coercion}\\
\text{Elim Terms} & E & ::= & u & \text{Hypotheses}\\
& & & \mid E\, I & \text{Implication}\\
& & & \mid \mathrm{fst}\, E \mid \mathrm{snd}\, E & \text{Conjunction}\\
& & & \mid (I : A) & \text{Coercion}
\end{array}
$$

The presence of E as an intro term corresponds to the coercion $\downarrow\Uparrow$ which is present in normal deductions. The presence of $(I : A)$ as an elim term corresponds to the coercion $\Uparrow\downarrow$ which is present only in the extended system. Therefore, a normal deduction can be represented without any internal type information, while a general deduction requires information at the point where an introduction rule is directly followed by an elimination rule. It is easy to endow the annotated natural deduction judgments with the modified proof terms from above. We leave the details to Exercise 3.3. The two judgments are $\Gamma^\downarrow \vdash^+ I : A \Uparrow$ and $\Gamma^\downarrow \vdash^+ E : A \downarrow$.

Now we can prove the correctness of bi-directional type-checking.

Theorem 3.4 (Bi-Directional Type-Checking)

1. *Given $\Gamma^\downarrow$, I, and A. Then either $\Gamma^\downarrow \vdash^+ I : A \Uparrow$ or not.*
2. *Given $\Gamma^\downarrow$ and E. Then either there is a unique A such that $\Gamma^\downarrow \vdash^+ E : A$ or there is no such A.*

Proof: See Exercise 3.3. □

3.3 Exercises

Exercise 3.1 Consider a system of normal deduction where the elimination rules for disjunction and existential are allowed to end in an extraction judgment.

$$\frac{\Gamma^{\downarrow} \vdash A \vee B \downarrow \qquad \Gamma^{\downarrow}, u{:}A \downarrow \vdash C \downarrow \qquad \Gamma^{\downarrow}, w{:}B \downarrow \vdash C \downarrow}{\Gamma^{\downarrow} \vdash C \downarrow} \vee\mathrm{E}^{u,w}$$

$$\frac{\Gamma^{\downarrow} \vdash \exists x.\, A \downarrow \qquad \Gamma^{\downarrow}, u{:}[a/x]A \downarrow \vdash C \downarrow}{\Gamma^{\downarrow} \vdash C \downarrow} \exists\mathrm{E}^{a,u}$$

Discuss the relative merits of allowing or disallowing these rules and show how they impact the subsequent development in this Chapter (in particular, bi-directional type-checking and the relationship to the sequent calculus).

Exercise 3.2

1. Give an example of a natural deduction which is *not* normal (in the sense defined in Section 3.1), yet contains no subderivation which can be locally reduced.
2. Generalizing from the example, devise additional rules of reduction so that any natural deduction which is not normal can be reduced. You should introduce no more and no fewer rules than you need for this purpose.
3. Prove that your rules satisfy the specification in part (2).

Exercise 3.3 Write out the rules defining the judgments $\Gamma^{\downarrow} \vdash^{+} I : A \Uparrow$ and $\Gamma^{\downarrow} \vdash^{+} E : A \downarrow$ and prove Theorem 3.4. Make sure to carefully state the induction hypothesis (if it is different from the statement of the theorem) and consider all the cases.