Chapter 5

The Inverse Method

After the definition of logic via natural deduction, we have developed a succession of techniques for theorem proving based on sequent calculi. We considered a sequent $\Gamma \Longrightarrow C$ as a goal, to be solved by backwards-directed search which was modeled by the bottom-up construction of a derivation. The critical choices were disjunctive non-determinism (resolved by guessing and backtracking) and existential non-determinism (resolved by introducing existential variables and unification). The limiting factor in more refined theorem provers based on this method is generally the number of disjunctive choices which have to be made. It is complicated by the fact that existential variables are global in a partial derivation, which means that choices in one conjunctive branch have effects in other branches. This effects redundancy elimination, since subgoals are not independent of each other.

The diametrically opposite approach would be to work forward from the initial sequents until the goal sequent is reached. If we guarantee a fair strategy in the selection of axioms and inference rules, every goal sequent can be derived this way. Without further improvements, this is clearly infeasible, since there are too many derivations for us to hope that we can find one for the goal sequent in this manner.

The *inverse method* is based on the property that in a cut-free derivation of a goal sequent, we only need to consider subformulas of the goal and their substitution instances. For example, when we have derived both A and B in the forward direction, we only derive their conjunction $A \wedge B$ if $A \wedge B$ is a subformula of the goal sequent.

The nature of forward search under these restrictions is quite different from the backward search. Since we always add new consequences to the sequents already derived, knowledge grows monotonically and no disjunctive non-determinism arises. Similarly for existential non-determinism, if we keep sequents in their maximally general form. On the other hand, there is a potentially very large amount of conjunctive non-determinism, since we have to apply all applicable rules to all sequents in a fair manner in order to guarantee completeness. The critical factor in forward search is to limit conjunctive non-determinism. We can view this as redundancy elimination: among the many ways that a given sequent may be derived, we try to actually consider a few as possible. The techniques developed in the preceding chapters, with some modifications, can be applied in this new setting.

Historically, the inverse method is due to Maslov [Mas64]. It has been adapted to intuitionistic and other non-classical logics by Voronkov [Vor92], Mints [Min94], and Tammet [Tam96, Tam97].

5.1 Forward Sequent Calculus

As a first step towards the inverse method, we write out a sequent calculus appropriate for forward search. This stems from a basic reinterpretation of a sequent during search. Previously, $\Gamma \Longrightarrow C$ expressed that we may use all hypotheses in Γ to prove that C is true. Now we will think of $\Gamma \longrightarrow C$ to mean that we needed all the hypotheses in Γ in order to prove that C is true.

This means that weakening is no longer valid for sequents $\Gamma \longrightarrow C$ and we have to take special care when we formulate correctness theorems. Secondly, we do not need to keep duplicate assumptions, so we view Γ in the sequent $\Gamma \longrightarrow C$ as a *set* of assumptions. We write $\Gamma_1 \cup \Gamma_2$ for the union of two sets of assumptions, and Γ, A stands for $\Gamma \cup \{A\}$.[1]

Initial Sequents. Previously, we allowed $\Gamma, A \Longrightarrow A$, since the assumptions in Γ can be used, but are just not needed in this case. In the forward calculus, initial sequents

$$\frac{}{A \longrightarrow A}\,\text{init}$$

express that only the hypothesis A is needed to derive the truth of A and nothing else.

Conjunction. In the right rule for conjunction, we previously concluded $\Gamma \Longrightarrow A \wedge B$ from $\Gamma \Longrightarrow A$ and $\Gamma \Longrightarrow B$. This expressed that all assumptions Γ are available in both branches. Now we need to take the union of the two sets of assumptions, expressing that both are needed to prove the conclusion.

$$\frac{\Gamma_1 \longrightarrow A \qquad \Gamma_2 \longrightarrow B}{\Gamma_1 \cup \Gamma_2 \longrightarrow A \wedge B}\,\wedge\text{R}$$

On the left rules, so such considerations arise.

$$\frac{\Gamma, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\,\wedge\text{L}_1 \qquad \frac{\Gamma, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\,\wedge\text{L}_2$$

Note that if $A \wedge B$ is already present in Γ in the two left rules, it will not be added again.

[1] In the language of judgments, $\Gamma \longrightarrow A$ is a *strict hypothetical judgment*.

Truth. As in the backward sequent calculus, there is only a right rule. Unlike the backward sequent calculus, it does not permit any hypotheses.

$$\frac{}{\cdot \longrightarrow \top}\,\top R$$

Implication. In the backward sequent calculus, the right rule for implication has the form

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\,{\supset}R.$$

In the forward direction this would not be sufficient, because it would allow us to conclude $A \supset B$ only if A is actually needed in the proof of B. To account for this case, we introduce two separate rules.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\,{\supset}R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \supset B}\,{\supset}R_2$$

Another, more efficient possibility combines these rules into one which removes A from the context of the premise if it is there and otherwise leaves it unchanged (see Section **??**). In the left rule we have to take a union as in the right rule for conjunction.

$$\frac{\Gamma_1 \longrightarrow A \qquad \Gamma_2, B \longrightarrow C}{\Gamma_1 \cup \Gamma_2, A \supset B \longrightarrow C}\,{\supset}L$$

Note that the principal proposition $A \supset B$ does not occur in the premises. However, it might occur in Γ_1 or Γ_2, in which case it is not added again in the conclusion.

Disjunction. This introduces no new considerations.

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\,{\vee}R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\,{\vee}R_2$$

$$\frac{\Gamma_1, A \longrightarrow C \qquad \Gamma_2, B \longrightarrow C}{\Gamma_1, \Gamma_2, A \vee B \longrightarrow C}\,{\vee}L$$

Falsehood. There is only a left rule.

$$\frac{}{\bot \longrightarrow C}\,{\bot}L$$

We postpone the consideration of negation and quantifiers.

The soundness of the forward sequent calculus is easy to establish.

Theorem 5.1
If $\Gamma \longrightarrow C$ *then* $\Gamma \Longrightarrow C$

Proof: By induction on the structure of the derivation $\mathcal{F}$ of $\Gamma \longrightarrow C$. We show only some of the cases, since the patterns are very similar in the remaining ones. In order to avoid confusion, we write Γ, A and $\Gamma \cup \{A\}$ for forward sequents to be more explicit about possible contractions.

Case:

$$\mathcal{F} = \frac{}{C \longrightarrow C}\,\text{init}$$

$C \Longrightarrow C$ By rule init

Case:

$$\mathcal{F} = \frac{\begin{array}{c}\mathcal{F}_1\\ \Gamma_1 \longrightarrow C_1\end{array} \qquad \begin{array}{c}\mathcal{F}_2\\ \Gamma_2 \longrightarrow C_2\end{array}}{\Gamma_1 \cup \Gamma_2 \longrightarrow C_1 \wedge C_2}\,\wedge\text{R}$$

$\Gamma_1 \Longrightarrow C_1$	By i.h. on $\mathcal{F}_1$
$\Gamma_1 \cup \Gamma_2 \Longrightarrow C_1$	By weakening
$\Gamma_2 \Longrightarrow C_2$	By i.h. on $\mathcal{F}_2$
$\Gamma_1 \cup \Gamma_2 \Longrightarrow C_2$	By weakening
$\Gamma_1 \cup \Gamma_2 \Longrightarrow C_1 \wedge C_2$	By rule $\wedge$R

Case:

$$\mathcal{F} = \frac{\begin{array}{c}\mathcal{F}_1\\ \Gamma_1 \longrightarrow A\end{array} \qquad \begin{array}{c}\mathcal{F}_2\\ \Gamma_2, B \longrightarrow C\end{array}}{\Gamma_1 \cup \Gamma_2 \cup \{A \supset B\} \longrightarrow C}\,\supset\text{L}$$

$\Gamma_1 \Longrightarrow A$	By i.h. on $\mathcal{F}_1$
$\Gamma_1 \cup \Gamma_2, A \supset B \Longrightarrow A$	By weakening
$\Gamma_2, B \Longrightarrow C$	By i.h. on $\mathcal{F}_2$
$\Gamma_1 \cup \Gamma_2, A \supset B, B \Longrightarrow C$	By weakening
$\Gamma_1 \cup \Gamma_2, A \supset B \Longrightarrow C$	By rule $\supset$L
$\Gamma_1 \cup \Gamma_2 \cup \{A \supset B\} \Longrightarrow C$	By contraction (if needed)

□

Completeness is more difficult. In fact, it is false! For example, for atomic propositions P and Q we can not derive $P, Q \Longrightarrow P$. Fortunately, the absence of weakening is the only source of difficulty and can easily be taken into account.

Theorem 5.2
If $\Gamma \Longrightarrow C$ *then* $\Gamma' \longrightarrow C$ *for some* $\Gamma' \subseteq \Gamma$.

Proof: By induction on the structure of $\mathcal{S}$ for $\Gamma \Longrightarrow C$.

Case:

$$\mathcal{S} = \frac{}{\Gamma_1, C \Longrightarrow C}\text{init}$$

$C \longrightarrow C$	By rule init
$\{C\} \subseteq \Gamma_1, C$	By definition of $\subseteq$

Case:

$$\mathcal{S} = \frac{\begin{array}{c}\mathcal{S}_1\\ \Gamma, A \Longrightarrow B\end{array}}{\Gamma \Longrightarrow A \supset B}\supset\!\text{R}$$

$\Gamma'' \longrightarrow B$ for some $\Gamma'' \subseteq \Gamma, A$	By i.h. on $\mathcal{S}_1$
$\Gamma'' = \Gamma', A$ and $\Gamma' \subseteq \Gamma$	First subcase
$\Gamma' \longrightarrow A \supset B$	By rule $\supset\!\text{R}_1$
$\Gamma'' \subseteq \Gamma$	Second subcase
$\Gamma'' \longrightarrow A \supset B$	By rule $\supset\!\text{R}_2$

Case:

$$\mathcal{S} = \frac{\begin{array}{c}\mathcal{S}_1\\ \Gamma_1, A \supset B \Longrightarrow A\end{array} \qquad \begin{array}{c}\mathcal{S}_2\\ \Gamma_1, A \supset B, B \Longrightarrow C\end{array}}{\Gamma_1, A \supset B \Longrightarrow C}\supset\!\text{L}$$

$\Gamma_1' \longrightarrow A$ for some $\Gamma_1' \subseteq \Gamma_1, A \supset B$	By i.h. on $\mathcal{S}_1$
$\Gamma_2' \longrightarrow C$ for some $\Gamma_2' \subseteq \Gamma_1, A \supset B, B$	By i.h. on $\mathcal{S}_2$
$\Gamma_2' = \Gamma_2'', B$ and $\Gamma_2'' \subseteq \Gamma_1, A \supset B$	First subcase
$\Gamma_1' \cup \Gamma_2'' \cup \{A \supset B\} \longrightarrow C$	By rule $\supset$L
$\Gamma_1' \cup \Gamma_2'' \cup \{A \supset B\} \subseteq \Gamma_1 \cup \{A \supset B\}$	By properties of $\subseteq$
$\Gamma_2' \subseteq \Gamma_1, A \supset B$	Second subcase
$\Gamma' = \Gamma_2'$ satisfies claim	

□

5.2 Negation and Empty Succedents

In the backward sequent calculus, the rules for negation

$$\frac{\Gamma, A \Longrightarrow p}{\Gamma \Longrightarrow \neg A}\neg\text{R}^p \qquad \frac{\Gamma, \neg A \Longrightarrow A}{\Gamma, \neg A \Longrightarrow C}\neg\text{L}$$

require propositional parameters p. In Gentzen's original formulation of the sequent calculus he avoided this complication by allowing an empty right-hand side. A sequent of the form

$$\Gamma \Longrightarrow \cdot$$

can then be interpreted as

$$\Gamma \Longrightarrow p \qquad \text{for a parameter } p \text{ not in } \Gamma$$

As a result we can substitute an arbitrary proposition for the right-hand side (the defining property for parametric judgments) and obtain an evident judgment. In the sequent calculus with empty right-hand sides, this can be accomplished by weakening on the right:

If $\Gamma \Longrightarrow \cdot$ then $\Gamma \Longrightarrow C$ for any proposition C.

When the right-hand side can be either empty or a singleton we write $\Gamma \Longrightarrow \gamma$.

In a forward sequent calculus we can take advantage of this in order to avoid overcommitment in the rules for negation and falsehood. We first show the forward rules for negation; then we reexamine all the previous rules.

Negation. We just take advantage of the new form of judgment, avoiding, for example, a commitment to a particular proposition C in the $\neg$L rule.

$$\frac{\Gamma, A \longrightarrow \cdot}{\Gamma \longrightarrow \neg A}\neg\mathrm{R} \qquad \frac{\Gamma \longrightarrow A}{\Gamma, \neg A \longrightarrow \cdot}\neg\mathrm{L}$$

Interestingly, we do not need a second right rule for negation as for implication (see Exercise **??**).

Falsehood. Falsehood can similarly benefit from avoiding commitment. Note that previously the rule stated $\bot \longrightarrow C$, although there is a lot of possible non-determinism for C. Now we just replace this by

$$\frac{}{\bot \longrightarrow \cdot}\bot\mathrm{L}$$

There still is no right rule.

Initial Sequents. They do not change.

$$\frac{}{A \longrightarrow A}\mathrm{init}$$

Conjunction. This requires no change, except that the right hand side might be empty.

$$\frac{\Gamma_1 \longrightarrow A \qquad \Gamma_2 \longrightarrow B}{\Gamma_1 \cup \Gamma_2 \longrightarrow A \wedge B}\wedge\mathrm{R}$$

On the left rules, so such considerations arise.

$$\frac{\Gamma, A \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma}\wedge\mathrm{L}_1 \qquad \frac{\Gamma, B \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma}\wedge\mathrm{L}_2$$

Truth. Does not change.

$$\frac{}{\cdot \longrightarrow \top}\,\top\text{R}$$

Implication. The possibility of empty right-hand sides requires a third right rule for implication. Again, in an implementation the three rules might be combined into a more efficient one.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\,{\supset}\text{R}_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \supset B}\,{\supset}\text{R}_2 \qquad \frac{\Gamma, A \longrightarrow \cdot}{\Gamma \longrightarrow A \supset B}\,{\supset}\text{R}_3$$

$$\frac{\Gamma_1 \longrightarrow A \qquad \Gamma_2, B \longrightarrow \gamma}{\Gamma_1 \cup \Gamma_2, A \supset B \longrightarrow \gamma}\,{\supset}\text{L}$$

Disjunction. The rule for disjunction on the right remains the same, but the left rule now has to account for several possibilities, depending on whether the right-hand sides are empty. Essentially, we take the union of the right-hand sides of the two premises, except that the result must be a singleton for the sequent to be well-formed.

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\,\vee\text{R}_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\,\vee\text{R}_2$$

$$\frac{\Gamma_1, A \longrightarrow \gamma_1 \qquad \Gamma_2, B \longrightarrow \gamma_2}{\Gamma_1, \Gamma_2, A \vee B \longrightarrow \gamma_1 \cup \gamma_2}\,\vee\text{L}$$

In detail, either γ_1 or γ_2 is empty, or $\gamma_1 = \gamma_2 = C = \gamma_1 \cup \gamma_2$. The rule does not apply otherwise.

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