Initial Sequents. This leaves the question of initial sequents, which is easily handled by allowing an left passive atomic proposition to match a right passive atomic proposition.

$$\frac{}{\Delta, P; \cdot \Longrightarrow \cdot; P}\,\text{init}$$

The judgments $\Delta; \Gamma \Longrightarrow \rho$ are hypothetical in Δ, but *not* hypothetical in Γ. This is because proposition in Γ do not persist, and because they have to be empty in the initial sequents. In other words, contraction and weakening are not available for Γ. However, it can be explained as a *linear hypothetical judgment* where each linear hypothesis must be used exactly once in a derivation. We do not formalize this notion any further, but just remark that appropriate versions of the substitution property can be devised to explain its meaning.

First, the soundness theorem is straightforward, since inversion proofs merely eliminate some disjunctive non-determinism.

Theorem 4.2 (Soundness of Inversion Proofs)
If $\Delta; \Gamma \Longrightarrow A; \cdot$ *or* $\Delta; \Gamma \Longrightarrow \cdot; A$ *then* $\Delta, \Gamma \Longrightarrow A$.

Proof: By a straightforward induction over the given derivation, applying weakening in some cases. □

Formulating appropriate theorems for the study of inversion proofs is somewhat difficult, because of the nature conjunctive and disjunctive non-determinism. To complement the soundness property, we first show the completeness theorem for the deductive system. Note, however, that this does not yet take into account the don't care non-determinism we have in mind for the sequents with active propositions.

The completeness theorem requires a number of lemmas about inversion sequents. For a possible alternative path, see Exercise 4.2. The first set of results expresses the invertibility of the rules concerning the active propositions. That is, we can immediately apply any invertible rule witout losing completeness. The second set of results expresses the opposite: we can always postpone the non-invertible rules until all invertible rules have been applied.

To state inversion in the strongest form (which is needed in the completeness proof for the search procedure, Theorem 4.8) we define the *depth* of a derivation as one plus the maximum of the depth of the derivations of the premises of the last rule applied. The depth is defined as 1 if the last inference rule has no premises.

Lemma 4.3 (Inversion on Active Rules)

1. *If* $\Delta; \Gamma \Longrightarrow A \wedge B; \cdot$ *then* $\Delta; \Gamma \Longrightarrow A$ *and* $\Delta; \Gamma \Longrightarrow A; \cdot$.
2. *If* $\Delta; \Gamma \Longrightarrow A \supset B; \cdot$ *then* $\Delta; \Gamma, A \Longrightarrow B; \cdot$.
3. *If* $\Delta; \Gamma \Longrightarrow \forall x.\, A; \cdot$ *then* $\Delta; \Gamma \Longrightarrow [a/x]A; \cdot$ *for any new parameter* a.

4. *If* $\Delta; \Gamma \Longrightarrow R; \cdot$ *then* $\Delta; \Gamma \Longrightarrow \cdot; R$.

5. *If* $\Delta; \Gamma, A \wedge B \Longrightarrow \rho$ *then* $\Delta; \Gamma, A, B \Longrightarrow \rho$.

6. *If* $\Delta; \Gamma, \top \Longrightarrow \rho$ *then* $\Delta; \Gamma \Longrightarrow \rho$.

7. *If* $\Delta; \Gamma, A \vee B \Longrightarrow \rho$ *then* $\Delta; \Gamma, A \Longrightarrow \rho$ *and* $\Delta; \Gamma, B \Longrightarrow \rho$.

8. *If* $\Delta; \Gamma, \exists x.\, A \Longrightarrow \rho$ *then* $\Delta; \Gamma, [a/x]A \Longrightarrow \rho$ *for any new parameter* a.

9. *If* $\Delta; \Gamma, L \Longrightarrow \rho$ *then* $\Delta, L; \Gamma \Longrightarrow \rho$.

Moreover, in each case the derivations whose existence is asserted are of equal or smaller depth than the given derivations.

Proof: By straightforward induction on the structure of the given derivations. □

The dual lemma shows that passive rules can be postponed until after the active rules. We define the *active size* of a sequent $\Delta; \Gamma \Longrightarrow A; \cdot$ or $\Delta; \Gamma \Longrightarrow \cdot; R$ as the number of logical quantifiers, connectives, constants, and atomic propositions in Γ and A. Note that the active size of a sequent is 0 if and only if it has the form $\Delta; \cdot \Longrightarrow \cdot; R$.

Lemma 4.4 (Postponement of Passive Rules)

1. *If* $\Delta; \Gamma \Longrightarrow A; \cdot$ *or* $\Delta; \Gamma \Longrightarrow \cdot; A$ *then* $\Delta; \Gamma \Longrightarrow \cdot; A \vee B$.

2. *If* $\Delta; \Gamma \Longrightarrow B; \cdot$ *or* $\Delta; \Gamma \Longrightarrow \cdot; B$ *then* $\Delta; \Gamma \Longrightarrow \cdot; A \vee B$.

3. *If* $\Delta; \Gamma \Longrightarrow [t/x]A; \cdot$ *or* $\Delta; \Gamma \Longrightarrow \cdot; [t/x]A$ *then* $\Delta; \Gamma \Longrightarrow \cdot; \exists x.\, A$.

4. *If* $(\Delta, A \supset B); \Gamma \Longrightarrow A; \cdot$ *and* $(\Delta, A \supset B); \Gamma, B \Longrightarrow \rho$
   then $(\Delta, A \supset B); \Gamma \Longrightarrow \rho$.

5. *If* $(\Delta, \forall x.\, A); \Gamma, [t/x]A \Longrightarrow \rho$ *then* $(\Delta, \forall x.\, A); \Gamma \Longrightarrow \rho$.

Proof: By induction on the active size of the given sequent. In the base case, the result follows directly by an inference rule. In each other case we apply inversion to an element of Γ (Lemma 4.3) and appeal to the induction hypothesis. We show two cases in the proof of part (4).

Case: $\Gamma = \cdot$ and $\rho = \cdot; R$.

$(\Delta, A \supset B); B \Longrightarrow \cdot; R$	Assumption
$(\Delta, A \supset B); \cdot \Longrightarrow A; \cdot$	Assumption
$(\Delta, A \supset B); \cdot \Longrightarrow \cdot; R$	By rule $\supset$L

Case: $\Gamma = \Gamma', C \wedge D$.

$(\Delta, A \supset B); \Gamma', C \vee D, B \Longrightarrow \rho$	Assumption
$(\Delta, A \supset B); \Gamma', C, B \Longrightarrow \rho$ and	
$(\Delta, A \supset B); \Gamma', D, B \Longrightarrow \rho$	By inversion
$(\Delta, A \supset B); \Gamma', C \vee D \Longrightarrow A; \cdot$	Assumption
$(\Delta, A \supset B); \Gamma', C \Longrightarrow A; \cdot$ and	
$(\Delta, A \supset B); \Gamma', D \Longrightarrow A; \cdot$	By inversion
$(\Delta, A \supset B); \Gamma', C \Longrightarrow \rho$	By i.h. on Γ', C
$(\Delta, A \supset B); \Gamma', D \Longrightarrow \rho$	By i.h. on Γ', D
$(\Delta, A \supset B); \Gamma', C \vee D \Longrightarrow \rho$	By rule $\vee$L

□

In sequent calculus, the main judgment $\Gamma \Longrightarrow A$ is hypothetical in Γ. This means Γ directly satisfies weakening and contraction (the additional substitution property is not relevant in this context). However, the inversion sequent $\Delta; \Gamma \Longrightarrow \rho$ is *not* hypothetical in Γ. In particular, weakening is not obvious (since Γ must be empty for a passive rule to apply) and contraction is not obvious (since elements of Γ are not propagated from the conclusion to the premises of the rules).

For the proof of completeness, and also to permit some optimizations in the search procedure, we need to show that weakening and contraction for propositions in Γ are admissible, at the price of possibly lengthening the derivation. Note that weakening and contraction for Δ is trivial, since inversion sequents *are* hypothetical in Δ.

Lemma 4.5 (Structural Properties of Inversion Sequents)

1. *If* $\Delta; \Gamma \Longrightarrow \rho$ *then* $(\Delta, A); \Gamma \Longrightarrow \rho$.
2. *If* $(\Delta, A, A); \Gamma \Longrightarrow \rho$ *then* $(\Delta, A); \Gamma \Longrightarrow \rho$.
3. *If* $\Delta; \Gamma \Longrightarrow \rho$ *then* $\Delta; (\Gamma, A) \Longrightarrow \rho$.
4. *If* $\Delta; (\Gamma, A, A) \Longrightarrow \rho$ *then* $\Delta; (\Gamma, A) \Longrightarrow \rho$.

Proof: Parts (1) and (2) follow as usual by straightforward structural inductions over the given derivations. Parts (3) and (4) follow by induction on the structure of A, taking advantage of the inversion properties for active propositions (Lemma 4.3) and parts (1) and (2) for passive propositions. □

The first completeness theorem below does not express the conjunctive non-determinism in the search for inversion proofs. This will be treated in a further refinement.

Theorem 4.6 (Completeness of Inversion Proofs)
If $\Gamma \Longrightarrow A$ *then* $\cdot; \Gamma \Longrightarrow A; \cdot$.

Proof: By induction on the structure of the given sequent derivation $\mathcal{S}$, taking advantage of the inversion, postponement, and structural properties proven in this section. We consider in turn: invertible right rules, invertible left rules, initial sequents, non-invertible right rules and non-invertible left rules.

Case:

$$\mathcal{S} = \dfrac{\begin{array}{cc} \mathcal{S}_1 & \mathcal{S}_2 \\ \Gamma \Longrightarrow A_1 & \Gamma \Longrightarrow A_2 \end{array}}{\Gamma \Longrightarrow A_1 \wedge A_2} \wedge\mathrm{R}$$

$\cdot;\Gamma \Longrightarrow A_1;\cdot$	By i.h. on $\mathcal{S}_1$
$\cdot;\Gamma \Longrightarrow A_2;\cdot$	By i.h. on $\mathcal{S}_2$
$\cdot;\Gamma \Longrightarrow A_1 \wedge A_2;\cdot$	By rule $\wedge$R

Cases: The right invertible rules $\supset$R and $\forall$R and also the case for $\top$R are similar to the case for $\wedge$R.

Case:

$$\mathcal{S} = \dfrac{\begin{array}{cc} \mathcal{S}_1 & \mathcal{S}_2 \\ \Gamma, B_1 \vee B_2, B_1 \Longrightarrow A & \Gamma, B_1 \vee B_2, B_2 \Longrightarrow A \end{array}}{\Gamma, B_1 \vee B_2 \Longrightarrow A} \vee\mathrm{L}$$

$\cdot;\Gamma, B_1 \vee B_2, B_1 \Longrightarrow A;\cdot$	By i.h. on $\mathcal{S}_1$
$\cdot;\Gamma, B_1 \vee B_2, B_2 \Longrightarrow A;\cdot$	By i.h. on $\mathcal{S}_2$
$\cdot;\Gamma, B_1 \vee B_2, B_1 \vee B_2 \Longrightarrow A;\cdot$	By rule $\vee$L
$\cdot;\Gamma, B_1 \vee B_2 \Longrightarrow A;\cdot$	By contraction (Lemma 4.5)

Cases: The left invertible rule $\exists$L and also the case for $\bot$L are similar to the case for $\vee$L.

Case:

$$\mathcal{S} = \dfrac{\begin{array}{c} \mathcal{S}_1 \\ \Gamma, B_1 \wedge B_2, B_1 \Longrightarrow A \end{array}}{\Gamma, B_1 \wedge B_2 \Longrightarrow A} \wedge\mathrm{L}_1$$

$\cdot;\Gamma, B_1 \wedge B_2, B_1 \Longrightarrow A;\cdot$	By i.h. on $\mathcal{S}_1$
$\cdot;\Gamma, B_1 \wedge B_2, B_1, B_2 \Longrightarrow A;\cdot$	By weakening (Lemma 4.5)
$\cdot;\Gamma, B_1 \wedge B_2, B_1 \wedge B_2 \Longrightarrow A$	By rule $\wedge$L
$\cdot;\Gamma, B_1 \wedge B_2 \Longrightarrow A$	By contraction (Lemma 4.5)

Case: The case for $\wedge\mathrm{L}_2$ is symmetric to $\wedge\mathrm{L}_1$. Note that there is no left rule for $\top$ in the sequent calculus, so the $\top$L rule on inversion sequents arises only from weakening (see the following case).

Case:

$$\mathcal{S} = \dfrac{}{\Gamma, P \Longrightarrow P}\,\text{init}$$

$P; \cdot \Longrightarrow \cdot; P$	By rule init
$P; \cdot \Longrightarrow P; \cdot$	By rule tR
$\cdot; P \Longrightarrow P; \cdot$	By rule tL
$\cdot; \Gamma, P \Longrightarrow P; \cdot$	By weakening (Lemma 4.5)

Case:

$$\mathcal{S} = \dfrac{\begin{array}{c}\mathcal{S}_1 \\ \Gamma \Longrightarrow A_1\end{array}}{\Gamma \Longrightarrow A_1 \vee A_2}\,\vee\mathrm{R}_1$$

$\cdot; \Gamma \Longrightarrow A_1; \cdot$	By i.h. on $\mathcal{S}_1$
$\cdot; \Gamma \Longrightarrow \cdot; A_1 \vee A_2$	By postponement (Lemma 4.4)
$\cdot; \Gamma \Longrightarrow A_1 \vee A_2; \cdot$	By rule tR

Cases: The cases for the non-invertible right rules $\vee\mathrm{R}_2$ and $\exists\mathrm{R}$ are similar to $\vee\mathrm{R}_1$.

Case:

$$\mathcal{S} = \dfrac{\begin{array}{c}\mathcal{S}_1 \\ \Gamma, B_1 \supset B_2 \Longrightarrow B_1\end{array} \qquad \begin{array}{c}\mathcal{S}_2 \\ \Gamma, B_1 \supset B_2, B_2 \Longrightarrow A\end{array}}{\Gamma, B_1 \supset B_2 \Longrightarrow A}\,{\supset}\mathrm{L}$$

$\cdot; \Gamma, B_1 \supset B_2 \Longrightarrow B_1; \cdot$	By i.h. on $\mathcal{S}_1$
$B_1 \supset B_2; \Gamma \Longrightarrow B_1; \cdot$	By inversion (Lemma 4.3)
$\cdot; \Gamma, B_1 \supset B_2, B_2 \Longrightarrow A; \cdot$	By i.h. on $\mathcal{S}_2$
$B_1 \supset B_2; \Gamma, B_2 \Longrightarrow A; \cdot$	By inversion (Lemma 4.3)
$B_1 \supset B_2; \Gamma \Longrightarrow A; \cdot$	By postponement (Lemma 4.4)
$\cdot; \Gamma, B_1 \supset B_2 \Longrightarrow A; \cdot$	By rule tL

Case: The cases for the non-invertible left rule $\forall\mathrm{L}$ is similar to $\supset\mathrm{L}$.

□

To capture the conjunctive non-determinism we think of an unproven sequent as a *goal* and the unproven leaves of a partially constructed derivation as *subgoals*. From the inversion properties for active propositions we already know that we do not lose completeness when applying active rules. However, it is conceivable that the eager application of active rules does not terminate, which means that the search procedure we have in mind would be incomplete. Fortunately, this is not the case. While the following termination property is not directly needed in the completeness proof for the search procedure, it foreshadows the argument used there.

Lemma 4.7 (Termination of Active Rules)
Given a goal $\Delta; \Gamma \Longrightarrow \rho$. Any sequence of applications of active rules terminates.

Proof: By induction on the active size of the given sequent. □

Next we describe a non-deterministic algorithm for proof search. There are a number of ways to eliminate the remaining disjunctive non-determinism. Typical is depth-first search, made complete by iterative deepening. The choice of the term t in the rules $\exists$R and $\forall$L is later solved by introducing free variables and equational constraints into the search procedures which are solved by unification (see Section **??**). Many futher refinements and improvements are possible on this procedures, but not discussed here.

Given a goal $\Delta; \Gamma \Longrightarrow \rho$.

1. If $\Gamma = \cdot$ and $\rho = \cdot; P$ succeed if P is in Δ.
2. If $\Gamma = \cdot$ and $\rho = \cdot; R$, but the previous case does not apply, guess an inference rule to reduce the goal. In the cases of $\exists$R and $\forall$L we also have to guess a term t. Solve each subgoal by recursively applying the procedure. This case represents a disjunctive choice (don't know non-determinism). If no rule applies, we fail.
3. If Γ is non-empty or $\rho = A; \cdot$, choose any active rule which applies and solve each of the subgoals by recursively applying the procedure. This represents a conjunctive choice (don't care non-determinism). Note that some active rule must always be applicable in this case.

This search procedure is clearly sound, because the inversion proof system is sound (Theorem 4.2). Furthermore, if there is a derivation the procedure will (in principle) always terminate and find some derivation if it guesses correctly in step (2).

Theorem 4.8 (Completeness of Inversion Search)
Given a goal $\Delta; \Gamma \Longrightarrow \rho$. If there is a derivation of the goal, the inversion search procedures terminates and finds a derivation for any choices made in step (3) and some choices made in step (2).

Proof: By nested induction on depth of the given derivation $\mathcal{I}$ and the active size of the given sequent. That is, we can apply the induction hypothesis if

1. the depth of the derivation $\mathcal{I}$ strictly decreases, or
2. the depth of $\mathcal{I}$ remains the same and the active size of the goal strictly decreases.

Case: $\mathcal{I}$ is an initial sequent. Then we are in situation (1) and succeed.

Case: $\mathcal{I}$ ends in a passive rule. Then we are in situation (2). We "guess" the same rule instance to reduce our goal. Each of the resulting subgoals now has a proof of strictly smaller depth, and we can apply the induction hypothesis.

Case: $\mathcal{I}$ ends in an active rule. In that case we are in situation (3). Independent of the last rule used in $\mathcal{I}$, we can apply any active rule to reduce our goal. By inversion (Lemma 4.3) each of the subgoals will have a proof of smaller or equal depth than $\mathcal{I}$. Moreover, the active size of the goal strictly decreases and we can apply the induction hypothesis to each subgoal.

□

4.2 Focusing

The search procedure based on inversion developed in the previous section still has an unacceptable amount of don't know non-determinism. The problem lies in the undisciplined use and proliferation of assumptions whose left rule is not invertible.

In a typical situation we have some universally quantified implications as assumptions. For example, Δ could be

$$\begin{array}{l}\forall x_1.\, \forall y_1.\, \forall z_1.\, P_1(x_1,y_1,z_1) \supset Q_1(x_1,y_1,z_1) \supset R_1(x_1,y_1,z_1),\\ \forall x_2.\, \forall y_2.\, \forall z_2.\, P_2(x_2,y_2,z_2) \supset Q_2(x_2,y_2,z_2) \supset R_2(x_2,y_2,z_2)\end{array}$$

If the right-hand side is passive, we now have to apply $\forall$L to one of the two assumptions. We assume we guess the first one and that we can guess an appropriate term t_1. After the $\forall$L rule and a left transition, we are left with

$$\begin{array}{l}\forall x_1.\, \forall y_1.\, \forall z_1.\, P_1(x_1,y_1,z_1) \supset Q_1(x_1,y_1,z_1) \supset R_1(x_1,y_1,z_1),\\ \forall x_2.\, \forall y_2.\, \forall z_2.\, P_2(x_2,y_2,z_2) \supset Q_2(x_2,y_2,z_2) \supset R_2(x_2,y_2,z_2),\\ \forall y_1.\, \forall z_1.\, P_1(t_1,y_1,z_1) \supset Q_1(t_1,y_1,z_1) \supset R_1(t_1,y_1,z_1).\end{array}$$

Again, we are confronted with a don't know non-deterministic choice, now between 3 possibilities. One can see that the number of possible choices quickly explodes. We can observe that the pattern above does not coincide with mathematical practice. Usually one applies an assumption or lemma of the form above by instantiating all the quantifiers and all preconditions at once. This strategy called *focusing* is a refinement of the inversion strategy presented in the previous section.

Roughly, when all propositions in a sequent are passive, we *focus* either on an assumption or the proposition we are trying to prove and then apply a sequence of non-invertible rules to the chosen proposition. This phase stops when either an invertible connective or an atomic proposition is reached.

As in the previous section, we capture this idea by a combination of a deductive system and a search strategy which distinguishes between conjunctive and disjunctive choices. We still use the sequents $\Delta; \Gamma \Longrightarrow A; \cdot$ and $\Delta; \Gamma \Longrightarrow \cdot; R$

with the same notation for simplicity. All the active rules (but not the initial sequents) are copied verbatim to this system. In addition, we need two new forms of sequents to express the focusing phase of proof search. We write

$\Delta; A \gg \cdot; R$	Focus on A on the left
$\Delta; \cdot \gg A; \cdot$	Focus on A on the right

The initial and passive rules of the inversion derivation are replaced by the following set of rules.

Decision. These rules decide which formula to focus on and are treated in a don't know non-deterministic manner. While we allow focusing on an atomic assumption, focusing on the succedent requires it to be non-atomic. The reason is our handling of initial sequents.

$$\frac{(\Delta, L); L \gg \cdot; R}{(\Delta, L); \cdot \Longrightarrow \cdot; R}\,\mathrm{dL} \qquad \frac{\Delta; \cdot \gg R^+; \cdot}{\Delta; \cdot \gg \cdot; R^+}\,\mathrm{dR}$$

Right Focus Propositions. The non-invertible rules on the right maintain the focus on principal formula of the inference. When we have reduced the right-hand side to a right-invertible or atomic proposition, we blur our focus and initiate decomposition with an active sequent. Here $\overline{R}$ is either P, $A \supset B$, $A \wedge B$, $\top$, and $\forall x.\, A$.

$$\frac{\Delta; \cdot \gg A; \cdot}{\Delta; \cdot \gg A \vee B; \cdot}\,\vee\mathrm{R}_1 \qquad \frac{\Delta; \cdot \gg B; \cdot}{\Delta; \cdot \gg A \vee B; \cdot}\,\vee\mathrm{R}_2$$

$$\text{no right focus rule for } \bot \qquad \frac{\Delta; \cdot \gg [t/x]A; \cdot}{\Delta; \cdot \gg \exists x.\, A; \cdot}\,\exists\mathrm{R}$$

$$\frac{\Delta; \cdot \Longrightarrow \overline{R}; \cdot}{\Delta; \cdot \gg \overline{R}; \cdot}\,\mathrm{bR}$$

Left Focus Propositions. The non-invertible rules on the left also maintain their focus on the principal formula of the inference. When we have reached a non-atomic left-invertible proposition, we blur our focus and initiate decomposition with an active sequent. Here $\overline{L}^+$ is either $A \vee B$, $\bot$, $\exists x.\, A$.

$$\frac{\Delta; B \gg \cdot; R \qquad \Delta; \cdot \Longrightarrow A; \cdot}{\Delta; A \supset B \gg \cdot; R}\,\supset\mathrm{L}$$

$$\frac{\Delta; A \gg \cdot; R}{\Delta; A \wedge B \gg \cdot; R}\,\wedge\mathrm{L}_1 \qquad \frac{\Delta; B \gg \cdot; R}{\Delta; A \wedge B \gg \cdot; R}\,\wedge\mathrm{L}_2$$

$$\text{no rule for } \top\mathrm{L} \qquad \frac{\Delta; \overline{L}^+ \Longrightarrow \cdot; R}{\Delta; \overline{L}^+ \gg \cdot; R}\,\mathrm{bL}$$

Note that the second premise of the $\supset$L rule is an unfocused sequent. From a practical point of view it is important to continue with the focusing steps in the first premise before attempting to prove the second premise, because the decomposition of B may ultimately fail when an atomic proposition is reached. Such a failure would render the possibly difficult proof of A useless.

It is possible to extend the definition of $\overline{L}^+$ to include conjunction and $\top$ and remove the left focus rules for conjunction. In some situations this would clearly lead to shorter proofs, but the present version appears to have less disjunctive non-determinism.[3]

Initial Sequents. There is a slight, but important asymmetry in the initial sequents: we require that we have focused on the left proposition.

$$\frac{}{\Delta; P \gg \cdot; P}\,\text{init}$$

Since this is the only rule which can be applied when the left focus formula is atomic, a proof attempt fails in a situation where $\Delta; P \gg \cdot; Q$ for $P \neq Q$. This is a very important property of the search, limiting non-determinism in focusing.

If one shows only applications of the decision rules in a derivation, the format is very close to *assertion-level proofs* as proposed by Huang [Hua94]. His motivation was the development of a formalism appropriate for the presentation of mathematical proofs in a human-readable form. This provides independent evidence for the value of focusing proofs. Focusing derivations themselves were developed by Andreoli [And92] in the context of classical linear logic. An adaptation to intuitionistic linear logic was given by Howe [How98] which is related the calculus LJT devised by Herbelin [Her95]. Herbelin's goal was to devise a sequent calculus whose derivations are in bijective correspondence to normal natural deductions. Due to the $\vee$, $\bot$ and $\exists$ elimination rules, this is not the case here.

The search procedure which works with focusing sequents is similar to the one for inversion: it mixes conjunctive non-determinism for active rules with disjunctive non-determinism for choice and focused rules. After the detailed development of inversion proofs, we will not repeat or extend the development here, but refer the interested reader to the literature. The techniques are very similar to the ones shown in Section 4.1.

4.3 Exercises

Exercise 4.1 Give an alternative proof of the inversion properties (Theorem 4.1) which does not use induction, but instead relies on admissibility of cut in the sequent calculus (Theorem 3.11).

[3] *[evaluate]*

Exercise 4.2 Formulate one or several cut rules directly on inversion sequents as presented in Section 4.1 and prove that they are admissible. Does this simplify the development of the completeness result for inversion proofs? Show how admissibility might be used, or illustrate why it is not much help.