Chapter 7

Equality

Reasoning with equality in first order logic can be accomplished *axiomatically.* That is, we can simply add reflexivity, symmetry, transitivity, and congruence rules for each predicate and function symbol and use the standard theorem proving technology developed in the previous chapters. This approach, however, does not take strong advantage of inherent properties of equality and leads to a very large and inefficent search space.

While there has been a deep investigation of equality reasoning in classical logic, much less is known for intuitionistic logic. Some recent references are [Vor96, DV99].

In this chapter we develop some of the techniques of equational reasoning, starting again from first principles in the definition of logic. We therefore recapitulate some of the material in earlier chapters, now adding equality as a new primitive predicate symbol.

7.1 Natural Deduction

We characterize equality by its introduction rule, which simply states that $s \doteq s$ for any term s.

$$\frac{}{\vdash s \doteq s} \doteq \mathrm{I}$$

We have already seen this introduction rule in unification logic in Section 4.3. In the context of unification logic, however, we did not consider hypothetical judgments, so we did not need or specify elimination rules for equality.

If we know $s \doteq t$ we can replace any number of occurrences of s in a true proposition and obtain another true proposition.

$$\frac{\vdash s \doteq t \qquad \vdash [s/x]A}{\vdash [t/x]A} \doteq \mathrm{E}_1$$

Symmetrically, we can also replaces of occurrences of t by s.

$$\frac{\vdash s \doteq t \qquad \vdash [t/x]A}{\vdash [s/x]A} \doteq \mathrm{E}_2$$

It might seem that this second rule is redundant, and in some sense it is. In particular, it is a *derivable* rule of the calculus with only $\doteq \mathrm{E}_1$:

$$\frac{\dfrac{\vdash s \doteq t \qquad \dfrac{}{\vdash s \doteq s} \doteq \mathrm{I}}{\vdash t \doteq s} \doteq \mathrm{E}_1 \qquad \vdash [t/x]A}{\vdash [s/x]A} \doteq \mathrm{E}_1$$

However, this deduction is not *normal* (as defined below), and without the second elimination rule the normalization theorem would not hold and cut elimination in the sequent calculus would fail. We continue this discussion below, after introducing normal derivations.

Next, we check the local soundness and completeness of the rules. First, local soundness:

$$\frac{\dfrac{}{\vdash s \doteq s} \doteq \mathrm{I} \qquad \begin{array}{c}\mathcal{D}\\ \vdash [s/x]A\end{array}}{\vdash [s/x]A} \doteq \mathrm{E}_1 \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ \vdash [s/x]A\end{array}$$

and the reduction for $\doteq \mathrm{E}_2$ is identical.

Second, we have to verify local completeness. There are two symmetric expansions

$$\begin{array}{c}\mathcal{D}\\ \vdash s \doteq t\end{array} \quad \Longrightarrow_E \quad \frac{\begin{array}{c}\mathcal{D}\\ \vdash s \doteq t\end{array} \qquad \dfrac{}{\vdash s \doteq s} \doteq \mathrm{I}}{\vdash s \doteq t} \doteq \mathrm{E}_1$$

and

$$\begin{array}{c}\mathcal{D}\\ \vdash s \doteq t\end{array} \quad \Longrightarrow_E \quad \frac{\begin{array}{c}\mathcal{D}\\ \vdash s \doteq t\end{array} \qquad \dfrac{}{\vdash t \doteq t} \doteq \mathrm{I}}{\vdash s \doteq t} \doteq \mathrm{E}_2$$

witnessing local completeness.

Note that the second is redundant in the sense that for local completeness we only need to show that there is *some* way to apply elimination rules so that we can reconstitute the connective by introduction rules. This is an interesting example where local completeness (in the absence of the $\doteq \mathrm{E}_2$ rule) does not imply global completeness.

Next we define normal and extraction derivations. These properties are given by the inherent role of introduction and elimination rules.

$$\frac{}{\vdash s \doteq s \Uparrow} \doteq \mathrm{I}$$

$$\frac{\vdash s \doteq t \downarrow \qquad \vdash [s/x]A \Uparrow}{\vdash [t/x]A \Uparrow} \doteq \mathrm{E}_1 \qquad \frac{\vdash s \doteq t \downarrow \qquad \vdash [t/x]A \Uparrow}{\vdash [s/x]A \Uparrow} \doteq \mathrm{E}_2$$

The elimination rule is similar to the rules for disjunction in the sense that there is a side derivation whose conclusion is copied from the premise to the conclusion of the elimination rule. In the case of disjunction, the copy is identical; here, some copies of s are replaced by t or vice versa.

Now we can see, why the derivation of $\doteq \mathrm{E}_2$ is not normal:

$$\frac{\dfrac{\vdash s \doteq t \downarrow \qquad \dfrac{}{\vdash s \doteq s \Uparrow} \doteq \mathrm{I}}{\vdash t \doteq s ?} \doteq \mathrm{E}_1 \qquad \vdash [t/x]A \Uparrow}{\vdash [s/x]A \Uparrow} \doteq \mathrm{E}_1$$

The judgment marked with ? should be $t \doteq s \Uparrow$ considering it is the conclusion of an equality elimination inference, and it should be $t \doteq s \downarrow$ considering it is the left premise of an equality elimination. Since no coercion from $\Uparrow$ to $\downarrow$ is available for normal derivations the deduction above cannot be annotated.

We assign proof terms only in their compact form (see Section 3.2). This means we have to analyse how much information is needed in the proof term to allow bi-directional type checking. Recall that we have introduction terms I and elimination terms E and that introduction terms are checked against a given type, while elimination term must carry enough information so that their type is unique. Following these considerations leads to the following new terms.

$$\begin{array}{llll} \text{Intro Terms} & I & ::= & \ldots \mid \mathrm{refl} & \text{for } \doteq \mathrm{I} \\ \text{Elim Terms} & E & ::= & \ldots \mid \mathrm{subst}_1^{\lambda x.A}\, E\, I & \text{for } \doteq \mathrm{E}_1 \\ & & & \mid \mathrm{subst}_2^{\lambda x.A}\, E\, I & \text{for } \doteq \mathrm{E}_2 \end{array}$$

The typing rules are straightforward. Recall that we localize the hypothesize to make the rules more explicit.

$$\frac{}{\Gamma^{\downarrow} \vdash \mathrm{refl} : s \doteq s \Uparrow} \doteq \mathrm{I}$$

$$\frac{\Gamma^{\downarrow} \vdash E : s \doteq t \downarrow \qquad \Gamma^{\downarrow} \vdash I : [s/x]A \Uparrow}{\Gamma^{\downarrow} \vdash \mathrm{subst}_1^{\lambda x.A}\, E\, I : [t/x]A \Uparrow} \doteq \mathrm{E}_1$$

$$\frac{\Gamma^{\downarrow} \vdash E : s \doteq t \downarrow \qquad \Gamma^{\downarrow} \vdash I : [t/x]A \Uparrow}{\Gamma^{\downarrow} \vdash \mathrm{subst}_2^{\lambda x.A}\, E\, I : [s/x]A \Uparrow} \doteq \mathrm{E}_2$$

We record the proposition A and an indication of the bound variable x in order to provide enough information for bi-direction type checking. Recall the desired property (Theorem 3.4):

1. *Given* $\Gamma^{\downarrow}$, I, *and* A. *Then either* $\Gamma^{\downarrow} \vdash I : A \Uparrow$ *or not.*
2. *Given* $\Gamma^{\downarrow}$ *and* E. *Then either there is a unique* A *such that* $\Gamma^{\downarrow} \vdash E : A \downarrow$ *or there is no such* A.

First, it is clear that the constant refl for equality introduction does not need to carry any terms, since $s \doteq s$ is given.

Second, to check $\mathrm{subst}_1^{\lambda x.A}\, E\, I$ against A' we first synthesize the type of E obtaining $s \doteq t$ and thereby s and t. Knowing t and A' does not determine A (consider, for example, $[t/x]A = q(t,t)$ which allows $A = q(x,x)$, $A = q(x,t)$, $A = q(t,x)$ and $A = q(t,t)$). However, A is recorded explicitly in the proof term, together with the variable x. Therefore we can now check whether the given type $[t/x]A$ is equal to A'. If that succeeds we have to check the introduction term I against $[s/x]A$ to verify the correctness of the whole term.

7.2 Sequent Calculus

The rules for the sequent calculus are determined by the definition of normal deduction as in Chapter 3. Introduction rules are turned into right rules; elimination rules into left rules.

$$\frac{}{\Gamma \Longrightarrow s \doteq s} \doteq \mathrm{R}$$

$$\frac{\Gamma, s \doteq t \Longrightarrow [s/x]A}{\Gamma, s \doteq t \Longrightarrow [t/x]A} \doteq \mathrm{L}_1 \qquad \frac{\Gamma, s \doteq t \Longrightarrow [t/x]A}{\Gamma, s \doteq t \Longrightarrow [s/x]A} \doteq \mathrm{L}_2$$

The proof for admissibility of cut in this calculus runs into difficulties when the cut formula was changed in the application of the $\doteq \mathrm{L}_1$ or $\doteq \mathrm{L}_2$ rules. Consider, for example, the cut between

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, s \doteq t \Longrightarrow [s/x]A\end{array}}{\Gamma, s \doteq t \Longrightarrow [t/x]A} \doteq \mathrm{L}_1 \qquad \text{and} \qquad \begin{array}{c}\mathcal{E}\\ \Gamma, s \doteq t, [t/x]A \Longrightarrow C\end{array}$$

If $[t/x]A$ is the principal formula of the last inference in $\mathcal{E}$, we would normally apply the induction hypothesis to $\mathcal{D}_1$ and $\mathcal{E}$, in effect pushing the cut past the last inference in $\mathcal{D}$. We cannot do this here, since $[s/x]A$ and $[t/x]A$ do not match. None of the rules in the sequent calculus without equality changed the conclusion in a left rule, so this situation did not arise before.

The simplest remedy seems to be to restrict the equality rules so they must be applied last in the bottom-up construction of a proof, and only to atomic formulas or other equalities. In this way, they cannot interfere with other inferences—they have been pushed up to the leaves of the derivation. This restriction is interesting for other purposes as well, since it allows us to separate equality reasoning from logical reasoning during the proof search process.

We introduce one new syntactice category and two new judgments. E stands for a *basic proposition*, which is either an atomic proposition P or an equation $s \doteq t$.

$$\begin{array}{ll} \Gamma \stackrel{E}{\Longrightarrow} E & E \text{ has an equational derivation from } \Gamma \\ \Gamma \stackrel{-}{\Longrightarrow} A & A \text{ has a regular derivation from } \Gamma \end{array}$$

Equational derivations are defined as follows.

$$\frac{}{\Gamma, P \stackrel{E}{\Longrightarrow} P}\,\text{init} \qquad \frac{}{\Gamma \stackrel{E}{\Longrightarrow} s \doteq s}\,\doteq\text{R}$$

$$\frac{\Gamma, s \doteq t \stackrel{E}{\Longrightarrow} [s/x]E}{\Gamma, s \doteq t \stackrel{E}{\Longrightarrow} [t/x]E}\,\doteq\text{L}_1 \qquad \frac{\Gamma, s \doteq t \stackrel{E}{\Longrightarrow} [s/x]E}{\Gamma, s \doteq t \stackrel{E}{\Longrightarrow} [t/x]E}\,\doteq\text{L}_1$$

Regular derivations have all the inference rules of sequent derivations without equality (except for initial sequents) plus the following coercion.

$$\frac{\Gamma \stackrel{E}{\Longrightarrow} E}{\Gamma \stackrel{-}{\Longrightarrow} E}\,\text{eq}$$

Regular derivations are sound and complete with respect to the unrestricted calculus. Soundness is direct.

Theorem 7.1 (Soudness of Regular Derivations)

1. *If* $\Gamma \stackrel{E}{\Longrightarrow} E$ *then* $\Gamma \Longrightarrow E$

2. *If* $\Gamma \stackrel{-}{\Longrightarrow} A$ *then* $\Gamma \Longrightarrow A$

Proof: By straightforward induction over the given derivations. □

In order to prove completeness we need a lemma which states that the unrestricted left equality rules are admissible in the restricted calculus. Because new assumptions are made, the statment of the lemma must actually be slightly more general by allowing substitution into hypotheses.

Lemma 7.2 (Admissibility of Generalized Equality Rules)

1. *If* $[s/x]\Gamma, s \doteq t \stackrel{-}{\Longrightarrow} [s/x]A$ *then* $[t/x]\Gamma, s \doteq t \stackrel{-}{\Longrightarrow} [t/x]A$.

2. *If* $[t/x]\Gamma, s \doteq t \stackrel{-}{\Longrightarrow} [t/x]A$ *then* $[s/x]\Gamma, s \doteq t \stackrel{-}{\Longrightarrow} [s/x]A$.

3. *If* $[s/x]\Gamma, s \doteq t \stackrel{E}{\Longrightarrow} [s/x]A$ *then* $[t/x]\Gamma, s \doteq t \stackrel{E}{\Longrightarrow} [t/x]A$.

4. *If* $[s/x]\Gamma, s \doteq t \stackrel{E}{\Longrightarrow} [s/x]A$ *then* $[t/x]\Gamma, s \doteq t \stackrel{E}{\Longrightarrow} [t/x]A$.

Proof: By induction on the structure of the given derivations $\mathcal{S}$ or $\mathcal{E}$, where the second and fourth parts are completely symmetric to the first and third part. In most cases this follows directly from the induction hypothesis. We show a few characteristic cases.

Case:

$$\mathcal{S} = \cfrac{\begin{matrix}\mathcal{S}_1 \\ [s/x]\Gamma, s \doteq t, [s/x]A_1 \overset{=}{\Longrightarrow} [s/x]A_2\end{matrix}}{[s/x]\Gamma, s \doteq t \overset{=}{\Longrightarrow} [s/x]A_1 \supset [s/x]A_2} \supset\mathrm{R}$$

$[t/x]\Gamma, s \doteq t, [t/x]A_1 \overset{=}{\Longrightarrow} [t/x]A_2$ By i.h. on $\mathcal{S}_1$

$[t/x]\Gamma, s \doteq t \overset{=}{\Longrightarrow} [t/x]A_1 \supset [t/x]A_2$ By rule $\supset$R

Case:

$$\mathcal{S} = \cfrac{\begin{matrix}\mathcal{E} \\ [s/x]\Gamma, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [s/x]E\end{matrix}}{[s/x]\Gamma, s \doteq t \overset{=}{\Longrightarrow} [s/x]E} \mathrm{eq}$$

$[t/x]\Gamma, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [t/x]E$ By i.h. (3) on $\mathcal{E}$

$[t/x]\Gamma, s \doteq t \overset{=}{\Longrightarrow} [t/x]E$ By rule eq

Case:

$$\mathcal{E} = \cfrac{}{[s/x]\Gamma', [s/x]P_1, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [s/x]P_2} \mathrm{init}$$

We obtain the first equation below from the assumption that $\mathcal{E}$ is an initial sequent.

$[s/x]P_1 = [s/x]P_2$ Given

$[t/x]\Gamma', [t/x]P_1, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [t/x]P_1$ By rule init

$[t/x]\Gamma', [t/x]P_1, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [s/x]P_1$ By rule $\doteq \mathrm{L}_2$

$[t/x]\Gamma', [t/x]P_1, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [s/x]P_2$ Same, by given equality

$[t/x]\Gamma', [t/x]P_1, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [t/x]P_2$ By rule $\doteq \mathrm{L}_1$

Case:

$$\mathcal{E} = \cfrac{\begin{matrix}\mathcal{E}' \\ [s/x]\Gamma', [s/x]q \doteq [s/x]r, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [[s/x]q/y]E'\end{matrix}}{[s/x]\Gamma', [s/x]q \doteq [s/x]r, s \doteq t \overset{\mathrm{E}}{\Longrightarrow} [s/x]E} \doteq \mathrm{L}_1$$

Note that we wrote the premise so that E' does contain an occurrence of x. We obtain the first equation below from the form of the inference rule $\doteq \mathrm{L}_1$.

$[s/x]E = [[s/x]r/y]E'$	Given
$[s/x]\Gamma', [s/x]q \doteq [s/x]r, s \doteq t \overset{E}{\Longrightarrow} [s/x][q/y]E'$	Same as $\mathcal{E}'$ (x not in E')
$[t/x]\Gamma', [t/x]q \doteq [t/x]r, s \doteq t \overset{E}{\Longrightarrow} [t/x][q/y]E'$	By i.h. on $\mathcal{E}'$
$[t/x]\Gamma', [t/x]q \doteq [t/x]r, s \doteq t \overset{E}{\Longrightarrow} [[t/x]q/y]E'$	Same, since x not in E'
$[t/x]\Gamma', [t/x]q \doteq [t/x]r, s \doteq t \overset{E}{\Longrightarrow} [[t/x]r/y]E'$	By rule $\doteq\mathrm{L}_1$
$[t/x]\Gamma', [t/x]q \doteq [t/x]r, s \doteq t \overset{E}{\Longrightarrow} [t/x][r/y]E'$	Same, since x not in E'
$[t/x]\Gamma', [t/x]q \doteq [t/x]r, s \doteq t \overset{E}{\Longrightarrow} [s/x][r/y]E'$	By rule $\doteq\mathrm{L}_2$
$[t/x]\Gamma', [t/x]q \doteq [t/x]r, s \doteq t \overset{E}{\Longrightarrow} [[s/x]r/y]E'$	Same, since x not in E'
$[t/x]\Gamma', [t/x]q \doteq [t/x]r, s \doteq t \overset{E}{\Longrightarrow} [s/x]E$	Same, by given equality
$[t/x]\Gamma', [t/x]q \doteq [t/x]r, s \doteq t \overset{E}{\Longrightarrow} [t/x]E$	By rule $\doteq\mathrm{L}_1$

Case:

$$\mathcal{E} = \cfrac{\begin{matrix}\mathcal{E}' \\ [s/x]\Gamma, s \doteq t \overset{E}{\Longrightarrow} [s/x]E'\end{matrix}}{[s/x]\Gamma, s \doteq t \overset{E}{\Longrightarrow} [s/x]E} \doteq\mathrm{L}_1$$

Note that we wrote the premise so that E' does contain an occurrence of x. We obtain the first line below from the shape of the conclusion in the inference rule $\doteq\mathrm{L}_1$ with the principal formula $s \doteq t$.

$[s/x]E = [t/x]E'$	Given
$[t/x]\Gamma, s \doteq t \overset{E}{\Longrightarrow} [t/x]E'$	By i.h. on $\mathcal{E}'$
$[t/x]\Gamma, s \doteq t \overset{E}{\Longrightarrow} [s/x]E$	Same, by given equality
$[t/x]\Gamma, s \doteq t \overset{E}{\Longrightarrow} [t/x]E$	By rule $\doteq\mathrm{L}_1$

□

A second lemma is helpful to streamline the completeness proof.

Lemma 7.3 (Atomic Initial Sequents) $\Gamma, A \overset{=}{\Longrightarrow} A$.

Proof: By induction on the structure of A. This is related to repeated local expansion. We show a few of cases.

Case: $A = P$.

$\Gamma, P \overset{E}{\Longrightarrow} P$	By rule init
$\Gamma, P \overset{=}{\Longrightarrow} P$	By rule eq

Case: $A = (s \doteq t)$.

$\Gamma, s \doteq t \overset{E}{\Longrightarrow} s \doteq s$	By rule $\doteq\mathrm{R}$
$\Gamma, s \doteq t \overset{E}{\Longrightarrow} s \doteq t$	By rule $\doteq\mathrm{L}_1$

Case: $A = A_1 \wedge A_2$.

$\Gamma, A_1 \overset{-}{\Longrightarrow} A_1$ By i.h. on A_1
$\Gamma, A_1 \wedge A_2 \overset{-}{\Longrightarrow} A_1$ By rule $\wedge L_1$
$\Gamma, A_2 \overset{-}{\Longrightarrow} A_2$ By i.h. on A_2
$\Gamma, A_1 \wedge A_2 \overset{-}{\Longrightarrow} A_2$ By rule $\wedge L_2$
$\Gamma, A_1 \wedge A_2 \overset{-}{\Longrightarrow} A_1 \wedge A_2$ By rule $\wedge R$

$\square$

With these two lemmas, completeness is relatively simple.

Theorem 7.4 (Completeness of Regular Derivations)
If $\Gamma \Longrightarrow A$ *then* $\Gamma \overset{-}{\Longrightarrow} A$.

Proof: By induction on the structure of the given derivation $\mathcal{S}$. We show some cases; most are straightforward.

Case:

$$\mathcal{S} = \cfrac{\begin{array}{c}\mathcal{S}_2\\ \Gamma, A_1 \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \supset A_2} \supset\!R$$

$\Gamma, A_1 \overset{-}{\Longrightarrow} A_2$ By i.h. on $\mathcal{S}_2$
$\Gamma \overset{-}{\Longrightarrow} A_1 \supset A_2$ By rule $\supset R$

Case:

$$\mathcal{S} = \cfrac{}{\Gamma', A \Longrightarrow A}\,\mathrm{init}$$

$\Gamma', A \overset{-}{\Longrightarrow} A$ By Lemma 7.3

Case:

$$\mathcal{S} = \cfrac{\begin{array}{c}\mathcal{S}_1\\ \Gamma', s \doteq t \Longrightarrow [s/x]A\end{array}}{\Gamma', s \doteq t \Longrightarrow [t/x]A} \doteq L_1$$

$\Gamma', s \doteq t \overset{-}{\Longrightarrow} [s/x]A$ By i.h. on $\mathcal{S}_1$
$\Gamma', s \doteq t \overset{-}{\Longrightarrow} [t/x]A$ By Lemma 7.2

$\square$

Regular derivations are the basis for proof search procedures. Furthermore, we can prove admissibility of cut, essentially following the same argument as in the system without equality for regular derivations. On equality derivations, we have to employ a new argument.

Theorem 7.5 (Admissibility of Cut with Equality)

1. *If* $\Gamma \stackrel{\mathrm{E}}{\Longrightarrow} E$ *and* $\Gamma, E \stackrel{\mathrm{E}}{\Longrightarrow} F$ *then* $\Gamma \stackrel{\mathrm{E}}{\Longrightarrow} F$.
2. *If* $\Gamma \stackrel{\mathrm{E}}{\Longrightarrow} E$ *and* $\Gamma, E \stackrel{=}{\Longrightarrow} C$ *then* $\Gamma \stackrel{=}{\Longrightarrow} C$.
3. *If* $\Gamma \stackrel{=}{\Longrightarrow} A$ *and* $\Gamma, A \stackrel{\mathrm{E}}{\Longrightarrow} F$ *then* $\Gamma \stackrel{=}{\Longrightarrow} F$.
4. *If* $\Gamma \stackrel{=}{\Longrightarrow} A$ *and* $\Gamma, A \stackrel{=}{\Longrightarrow} C$ *then* $\Gamma \stackrel{=}{\Longrightarrow} C$.

Proof: We prove the properties in sequence, using earlier ones to in the proofs of later ones.

Part (1). Given

$$\begin{array}{c}\mathcal{E}\\ \Gamma \stackrel{\mathrm{E}}{\Longrightarrow} E\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathcal{F}\\ \Gamma, E \stackrel{\mathrm{E}}{\Longrightarrow} F\end{array}$$

we construct a derivation for $\Gamma \stackrel{\mathrm{E}}{\Longrightarrow} F$ by nested induction on the structure of $\mathcal{E}$ and $\mathcal{F}$. That is, in appeals to the induction hypothesis, $\mathcal{E}$ may be smaller (in which case $\mathcal{F}$ may be arbitrary), or $\mathcal{E}$ stays the same and $\mathcal{F}$ gets smaller.

Cases: If E is a side formula of the last inference in $\mathcal{F}$ we appeal to the induction hypothesis on the premise and reapply the inference on the result. If $\mathcal{F}$ is an initial sequent we can directly construct the desired derivation. In the remaining cases, we assume E is the principal formula of the last inference in $\mathcal{F}$.

Case:

$$\mathcal{E} = \frac{}{\Gamma \stackrel{\mathrm{E}}{\Longrightarrow} s \doteq s}\, \doteq\mathrm{R} \qquad \text{and} \qquad \mathcal{F} = \frac{\begin{array}{c}\mathcal{F}_1\\ \Gamma, s \doteq s \stackrel{\mathrm{E}}{\Longrightarrow} [s/x]F_1\end{array}}{\Gamma, s \doteq s \stackrel{\mathrm{E}}{\Longrightarrow} [s/x]F_1}\, \doteq\mathrm{L}_1$$

$\Gamma \Longrightarrow [s/x]F_1$ By i.h. on $\mathcal{E}$ and $\mathcal{F}_1$

Case:

$$\mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1\\ \Gamma', q \doteq r \stackrel{\mathrm{E}}{\Longrightarrow} [q/x]s' = [q/x]t'\end{array}}{\Gamma', q \doteq r \stackrel{\mathrm{E}}{\Longrightarrow} [r/x]s' \doteq [r/x]t'}\, \doteq\mathrm{L}_1$$

$\Gamma', q \doteq r, [r/x]s' \doteq [r/x]t' \stackrel{\mathrm{E}}{\Longrightarrow} F$ $\mathcal{F}$, in this case
$\Gamma', q \doteq r, [q/x]s' \doteq [q/x]t' \stackrel{\mathrm{E}}{\Longrightarrow} F$ By Lemma 7.2
$\Gamma', q \doteq r \stackrel{\mathrm{E}}{\Longrightarrow} F$ By i.h. on $\mathcal{E}_1$ and above

Part (2) : Given

$$\begin{array}{c}\mathcal{E}\\ \Gamma \overset{\mathrm{E}}{\Longrightarrow} E\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathcal{S}\\ \Gamma, E \overset{=}{\Longrightarrow} C\end{array}$$

we construct a derivation for $\Gamma \overset{=}{\Longrightarrow} C$ by induction over the structure of $\mathcal{S}$. Since E is either atomic or an equality, it cannot be the principal formula of an inference in $\mathcal{S}$. When we reach a coercion from $\overset{\mathrm{E}}{\Longrightarrow}$ to $\overset{=}{\Longrightarrow}$ in $\mathcal{S}$ we appeal to Part (1).

Part (3) : Given

$$\begin{array}{c}\mathcal{S}\\ \Gamma \overset{=}{\Longrightarrow} A\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathcal{F}\\ \Gamma, A \overset{\mathrm{E}}{\Longrightarrow} F\end{array}$$

we construct a derivation for $\Gamma \overset{\mathrm{E}}{\Longrightarrow} F$ by nested induction on the structure of $\mathcal{F}$ and $\mathcal{S}$. If A is the principal formula of an inference in $\mathcal{F}$ then A must be atomic or an equality. In the former case we can derive the desired conclusion directly; in the latter case we proceed by induction over $\mathcal{S}$. Since A is an equality, it cannot be the principal formula of an inference in $\mathcal{S}$. When we reach a coercion for $\overset{\mathrm{E}}{\Longrightarrow}$ to $\overset{=}{\Longrightarrow}$ in $\mathcal{S}$ we appeal to Part (1).

Part (4) : Given

$$\begin{array}{c}\mathcal{S}\\ \Gamma \overset{=}{\Longrightarrow} A\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathcal{T}\\ \Gamma, A \overset{=}{\Longrightarrow} C\end{array}$$

we construct a derivation for $\Gamma \overset{=}{\Longrightarrow} C$ by nested induction on the structure of A, and the derivations $\mathcal{S}$ and $\mathcal{T}$ as in the proof of admissibility of cut without equality (Theorem 3.11). When we reach coercions from equality derivations we appeal to Parts 3 or 2. □

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