3.4 Inversion for Unrestricted Resources

Inversion principles as presented in Section 3.1 reduce don't-know non-deterministic choices by giving us license to always apply strongly invertible rules in the bottom-up search for a derivation. But there is a gap in the analysis in that the dereliction rule is *always* applicable when we have any unrestricted hypotheses, but is not invertible. However, there are a number of cases where we can write out derived or admissible rules that operate directly on unrestricted hypotheses, and which are invertible. We can then limit the use of dereliction to the remaining cases.

The following rules are all admissible and strongly invertible.

$$\frac{(\Gamma, A_1, A_2); \Delta \stackrel{-}{\Longrightarrow} B}{(\Gamma, A_1 \& A_2); \Delta \stackrel{-}{\Longrightarrow} B}\,\&\mathrm{L}! \qquad \frac{\Gamma; \Delta \stackrel{-}{\Longrightarrow} B}{(\Gamma, \top); \Delta \stackrel{-}{\Longrightarrow} B}\,\top\mathrm{L}!$$

$$\frac{\Gamma; \Delta \stackrel{-}{\Longrightarrow} B}{(\Gamma, \mathbf{1}); \Delta \stackrel{-}{\Longrightarrow} B}\,\mathbf{1}\mathrm{L}! \qquad \frac{}{(\Gamma, \mathbf{0}); \Delta \stackrel{-}{\Longrightarrow} B}\,\mathbf{0}\mathrm{L}!$$

$$\frac{(\Gamma, A); \Delta \stackrel{-}{\Longrightarrow} B}{(\Gamma, !A); \Delta \stackrel{-}{\Longrightarrow} B}\,!\mathrm{L}!$$

Theorem 3.13 (Invertibility of Admissible Left! Rules) *The rules* $\&\mathrm{L}!$, $\top\mathrm{L}!$, $\mathbf{0}\mathrm{L}!$ *and* $!\mathrm{L}!$ *are admissible and invertible. A system with these rules and dereliction restricted to a principal propositions which is of the form* P, $A_1 \multimap A_2$, $\forall x.\, A$, $A_1 \supset A_2$, $A_1 \otimes A_2$, $A_1 \oplus A_2$, *or* $\exists x.\, A$ *is sound and complete.*

Proof: Admissibility and invertibility follows direct calculation in each direction, using the admissibility of Cut! (Theorem **??**) in some cases. Soundness follows easily from admissibility, completeness from invertibility. □

There is also one derivable, weakly invertible rule.

$$\frac{}{(\Gamma, P); \cdot \stackrel{-}{\Longrightarrow} P}\,\mathrm{I}!$$

None of the remaining connectives admit invertible rules of the kind above (see Exercise **??**). If we want a complete system of rules to replace dereliction DL altogether, we would have to add some non-invertible ones. Here is a possible set of rules.

$$\frac{(\Gamma, A_1 \multimap A_2); \Delta_1 \overset{-}{\Longrightarrow} A_1 \qquad (\Gamma, A_1 \multimap A_2); (\Delta_2, A_2) \overset{-}{\Longrightarrow} B}{(\Gamma, A_1 \multimap A_2); \Delta_1 \times \Delta_2 \overset{-}{\Longrightarrow} B} \multimap \mathrm{L}!$$

$$\frac{(\Gamma, A_1 \supset A_2); \cdot \overset{-}{\Longrightarrow} A_1 \qquad (\Gamma, A_2); \Delta \overset{-}{\Longrightarrow} B}{(\Gamma, A_1 \multimap A_2); \Delta \overset{-}{\Longrightarrow} B} \multimap \mathrm{L}!$$

$$\frac{(\Gamma, \forall x.\ A); (\Delta, [t/x]A) \overset{-}{\Longrightarrow} B}{(\Gamma, \forall x.\ A); \Delta \overset{-}{\Longrightarrow} B} \forall \mathrm{L}! \qquad \frac{(\Gamma, \exists x.\ A); (\Delta, [a/x]A) \overset{-}{\Longrightarrow} B}{(\Gamma, \exists x.\ A); \Delta \overset{-}{\Longrightarrow} B} \exists \mathrm{L}!$$

$$\frac{(\Gamma, A_1 \otimes A_2); (\Delta, A_1, A_2) \overset{-}{\Longrightarrow} B}{(\Gamma, A_1 \otimes A_2); \Delta \overset{-}{\Longrightarrow} B} \otimes \mathrm{L}!$$

$$\frac{(\Gamma, A_1 \oplus A_2); (\Delta, A_1) \overset{-}{\Longrightarrow} B \qquad (\Gamma, A_1 \oplus A_2); (\Delta, A_2) \overset{-}{\Longrightarrow} B}{(\Gamma, A_1 \oplus A_2); \Delta \overset{-}{\Longrightarrow} B} \oplus \mathrm{L}!$$

In some cases there are other admissible rules, but they are rarely useful. For example, the rule

$$\frac{(\Gamma, \forall x.\ A, [t/x]A); \Delta \overset{-}{\Longrightarrow} B}{(\Gamma, \forall x.\ A); \Delta \overset{-}{\Longrightarrow} B} \forall \mathrm{L}!'$$

is certainly admissible and even invertible, but it cannot be applied eagerly, since it would lead to non-termination. Instead, we can simply reuse $\forall x.\ A$ if we need another copy of $[t/x]A$.

3.5 Another Example: Arithmetic

Because linear hypotheses must be used exactly once, we can encode arithmetic problems as propositions in linear logic. We map a set of linear equations over the natural numbers into a proposition of linear logic, such that any proof of the proposition corresponds to a solution to the set of equations. When the proposition has not proof, the linear equations have no solutions.

We first represent natural numbers using a new (uninterpreted) atomic proposition p.

$$\begin{array}{rcl} \ulcorner 0 \urcorner & = & \mathbf{1} \\ \ulcorner n+1 \urcorner & = & p \otimes \ulcorner n \urcorner \end{array}$$

Since $p \otimes \mathbf{1} \dashv\vdash p$ we omit the trailing $\mathbf{1}$ in the examples, and also sometimes abbreviate $\ulcorner n \urcorner$ as p^n.

Addition is then easily represented by the multiplicative conjunction, and equality by linear implication. We use e to range over arithmetic expressions (which are not yet completely defined).

$$\begin{array}{rcl} \ulcorner e_1 + e_2 \urcorner & = & \ulcorner e_1 \urcorner \otimes \ulcorner e_2 \urcorner \\ \ulcorner e_1 = e_2 \urcorner & = & \ulcorner e_1 \urcorner \multimap \ulcorner e_2 \urcorner \end{array}$$

For example, the equation $3 + 2 = 1 + 4$ would be represented as

$$(p \otimes p \otimes p \otimes \mathbf{1}) \otimes (p \otimes p \otimes \mathbf{1}) \multimap (p \otimes \mathbf{1}) \otimes (p \otimes p \otimes p \otimes p \otimes \mathbf{1})$$

which is clearly true. It is also easy to see that an equation between different numbers will be an unprovable linear implication.

For every variable x in the left-hand side of an equation we have a hypothesis $!p$. If the variable x is instantiated by a number n, the corresponding derivation will use this hypothesis n times, creating a linear copy of p each time. For example (omitting $\mathbf{1}$s):

$$\ulcorner x + y + 1 = 3 \urcorner = !p \otimes !p \otimes p \multimap p \otimes p \otimes p$$

If a variable x occurs more than once, or multiplied by a constant, we collect common terms and think of $kx = x + x + \cdots + x$. So the representation of $3x$ is $!(p \otimes p \otimes p)$. For example,

$$\ulcorner 3x + 2y = 7 \urcorner = !(p^3) \otimes !(p^2) \multimap p^7$$

Representing several simultaneous equations is a bit more difficult, because we must make sure that a variable is instantiated to the same number in all equations. We achieve this by using a different representation of the natural numbers in each equation (say p_i for equation number i), and let each variable generate the appropriate number of p_i's for equation i. The right-hand sides of the equations are then combined with $\otimes$. For example,

$$\begin{array}{ll} \ulcorner x + y + 1 = 4 \wedge 2x + 3y = 6 \urcorner = & \\ !(p_1 \otimes p_2^2) & x \text{ and } 2x \\ \otimes !(p_1 \otimes p_2^3) & y \text{ and } 3y \\ \multimap & \\ (p_1 \multimap p_1^4) & (x + y) + 1 = 4 \\ \otimes p_2^6 & \text{and } (2x + 3y) = 6 \end{array}$$

The multiplicative conjunction in the conclusion forces all hypotheses regarding p_1 to the first conjunct, and all hypothesis regarding p_2 to the second conjunct. Since the exponential ! operator is outside the tensor for the variables in the different equations, the number of uses of this unrestricted assumption determines the instantiation for the variable.

Note that this example requires only a very small fragment, the so-called *multiplicative exponential linear logic.*

$$\textit{Multiplicative Exponential} \quad M ::= \quad P \mid M_1 \otimes M_2 \mid \mathbf{1} \mid M_1 \multimap M_2 \mid !M$$

Actually, in the propositions above a linear implication never appears on the left-hand side of a linear implication, and an exponential never appears on the right-hand side of a linear implication, which is a significant further restriction.[3]

One can add negative numbers and stay within the multiplicative exponential fragment although it seems one left-nested implication is now necessary. For each equation i we add a new propositional constant q_i representing -1, and the hypothesis

$$!((p_i \otimes q_i) \multimap \mathbf{1})$$

expressing that $1 + (-1) = 0$. An occurrence of a variable on the right-hand side of an equation is implemented as a negative occurrence on the left. For example,

$\ulcorner x + 1 = 2y \wedge 2x + y - 2 = 3 \urcorner =$	
$!(p_1 \otimes q_1 \multimap \mathbf{1})$	1+(-1)=0
$\otimes!(p_2 \otimes q_2 \multimap \mathbf{1})$	1+(-1)=0
$\otimes!(p_1 \otimes p_2^2)$	x and $2x$
$\otimes!(q_1^2 \otimes p_2)$	$-2y$ and y
$\multimap$	
$(p_1 \multimap \mathbf{1})$	$(x - 2y) + 1 = 0$
$\otimes(q_2^2 \multimap p_2^3)$	and $(2x + y) - 2 = 3$

[3] *[add a note on what is known about the complexity of these two fragments]*