3.3 Resource Management

A form of choice unique to linear logic proof search is *resource management*: in the bottom-up application of the left rule for implication and right rule for tensor, we have to split the linear hypotheses and distribute them to the premisses. We would like to postpone this choice until the further structure of the derivation provides hints which resources might be needed in which subgoals.

To resolve this non-determinism, we use a technique inspired by unification. We pass the complete list of hypotheses to both premisses and maintain constraints which express that each hypothesis must be used in one of the two subderivations, but not both. If one is ever used in one branch, we can propagate this information to the other branch by constraint simplification. This mirrors the way unification propagates substitutions for existential variables between incomplete proof branches.

We annotate each hypothesis with an *occurrence label* b.

$$\textit{Occurrence Labels} \quad b \quad ::= \quad \top \mid \bot \mid o$$

Here, $\top$ labels a hypothesis which is definitely present and must therefore be consumed (in the bottom-up search), $\bot$ labels a hypothesis which is definitely not present and can therefore not be used, and an occurrence variable o labels a hypothesis which may or may not be used, subject to some global constraints. Constraints which arise all have the following forms.

$$\textit{Occurrence Constraints} \quad c \quad ::= \quad b_1 \doteq b_2 \mid b_1 + b_2 \doteq b_3 \mid c_1 \wedge c_2 \mid \mathrm{tt}$$

The validity judgment for constraints, $\models c$, is defined by the following rules.

$$\frac{}{\models \top \doteq \top} \doteq\top \qquad \frac{}{\models \bot \doteq \bot} \doteq\bot$$

$$\frac{}{\models \top + \bot \doteq \top} +\top\bot \qquad \frac{}{\models \bot + \top \doteq \top} +\bot\top$$

$$\textit{no } {+}\top\top \textit{ rule} \qquad \frac{}{\models \bot + \bot \doteq \bot} +\bot\bot$$

$$\frac{\models c_1 \qquad \models c_2}{\models c_1 \wedge c_2} \wedge\mathrm{i} \qquad \frac{}{\models \mathrm{tt}} \mathrm{tti}$$

We say a constraint c with occurrence variables is *satisfiable* if there is an assignment of $\top$ and $\bot$ to the occurrence variables such that the resulting constraint is valid.

Linear hypotheses, annotated with occurrence labels, have the form

$$\textit{Annotated Contexts} \quad \Delta \quad ::= \quad \cdot \mid \Delta, w^b{:}A$$

It is convenient to abbreviate $w^b{:}A$ as A^b and $\cdot, w_1^{b_1}{:}A_1, \ldots, w_n^{b_n}{:}A_n$ as $\Delta^{\vec{b}}$. The basic sequent now reads $\Gamma; \Delta^{\vec{b}} \Longrightarrow C \setminus c$, where c are the residual constraints. The intuition should be that any satisfying assignment to the occurrence variables in c leads to a valid derivation of $\Gamma; \Delta^* \Longrightarrow C$, where Δ^* retains hypotheses of the form $w^\top{:}A$ and erases hypotheses of the form $w^\perp{:}A$. We now go through the rules, removing resource non-determinism in favor of occurrence constraints. In practice, these constraint should be checked for satisfiability in each step for early detection of failure. To give a more compact presentation of the rules, we further write

$$\begin{array}{rll} \vec{b} \doteq \perp & \text{for} & b_1 \doteq \perp \wedge \cdots \wedge b_n \doteq \perp \quad \text{and} \\ \vec{o'} + \vec{o''} \doteq \vec{b} & \text{for} & o_1' + o_1'' = b_1 \wedge \cdots \wedge o_n' + o_n'' \doteq b_n. \end{array}$$

Hypotheses. Initial sequents change form, since the particular hypothesis we use must be constraint to be present, while all others have to be constrained to be absent. This leaves some residual non-determinism if several available hypotheses match the conclusion.

$$\frac{}{\Gamma; (\Delta^{\vec{b}}, A^d) \Longrightarrow A \setminus d \doteq \top \wedge \vec{b} \doteq \perp}\,\mathrm{I} \qquad \frac{(\Gamma, A); (\Delta, A^\top) \Longrightarrow C}{(\Gamma, A); \Delta \Longrightarrow C}\,\mathrm{DL}$$

Multiplicative Connectives. Multiplicative connectives have to generate constraints as discussed above. New linear hypothesis must be used somewhere, so their initial annotation is $\top$.

$$\frac{\Gamma; \Delta^{\vec{b}}, A^\top \Longrightarrow B \setminus c}{\Gamma; \Delta^{\vec{b}} \Longrightarrow A \multimap B \setminus c}\,{\multimap}\mathrm{R}$$

Whenever a left rule is applied to a hypothesis, its occurrence label is constrained to the $\top$. In addition, since linear implication is multiplicative, we generate *new* occurrence variables $\vec{o'}$ and $\vec{o''}$ and constrain them.

$$\frac{\Gamma; \Delta^{\vec{o'}} \Longrightarrow A \setminus c' \qquad \Gamma; \Delta^{\vec{o''}}, B \Longrightarrow C \setminus c''}{\Gamma; \Delta^{\vec{b}}, (A \multimap B)^d \Longrightarrow C \setminus d \doteq \top \wedge c' \wedge c'' \wedge \vec{o'} + \vec{o''} \doteq \vec{b}}\,{\multimap}\mathrm{L}$$

The tensor rules are similar. Here, too, the occurrence variables $\vec{o'}$ and $\vec{o''}$ must be new.

$$\frac{\Gamma; \Delta^{\vec{o'}} \Longrightarrow A \setminus c' \qquad \Gamma; \Delta^{\vec{o''}} \Longrightarrow B \setminus c''}{\Gamma; \Delta^{\vec{b}} \Longrightarrow A \otimes B \setminus c' \wedge c'' \wedge \vec{o'} + \vec{o''} \doteq \vec{b}}\,{\otimes}\mathrm{R}$$

$$\frac{\Gamma; \Delta^{\vec{b}}, A^\top, B^\top \Longrightarrow C \setminus c}{\Gamma; \Delta^{\vec{b}}, (A \otimes B)^d \Longrightarrow C \setminus d \doteq \top \wedge c}\,{\otimes}\mathrm{L}$$

The **1**R rule permits no linear hypotheses, so all of them are constrained to be absent.

$$\frac{}{\Gamma; \Delta^{\vec{b}} \Longrightarrow \mathbf{1} \setminus \vec{b} \doteq \bot}\,\mathbf{1}\mathrm{R} \qquad \frac{\Gamma; \Delta^{\vec{b}} \Longrightarrow C \setminus c}{\Gamma; \Delta^{\vec{b}}, \mathbf{1}^d \Longrightarrow C \setminus d \doteq \top \wedge c}\,\mathbf{1}\mathrm{L}$$

Additive Connectives. The additive connective are much simpler and do not affect the occurrence constraints, except that the principal proposition of a left rule must be constrained to be present.

$$\frac{\Gamma; \Delta^{\vec{b}} \Longrightarrow A \setminus c' \qquad \Gamma; \Delta^{\vec{b}} \Longrightarrow B \setminus c''}{\Gamma; \Delta^{\vec{b}} \Longrightarrow A \& B \setminus c' \wedge c''}\,\&\mathrm{R}$$

$$\frac{\Gamma; \Delta^{\vec{b}}, A^{\top} \Longrightarrow C \setminus c}{\Gamma; \Delta^{\vec{b}}, (A \& B)^d \Longrightarrow C \setminus d \doteq \top \wedge c}\,\&\mathrm{L}_1 \qquad \frac{\Gamma; \Delta^{\vec{b}}, B^{\top} \Longrightarrow C \setminus c}{\Gamma; \Delta^{\vec{b}}, (A \& B)^d \Longrightarrow C \setminus d \doteq \top \wedge c}\,\&\mathrm{L}_2$$

$$\frac{}{\Gamma; \Delta^{\vec{b}} \Longrightarrow \top \setminus \mathrm{tt}}\,\top\mathrm{R} \qquad \textit{No } \top \textit{ left rule}$$

$$\frac{\Gamma; \Delta^{\vec{b}} \Longrightarrow A \setminus c}{\Gamma; \Delta^{\vec{b}} \Longrightarrow A \oplus B \setminus c}\,\oplus\mathrm{R}_1 \qquad \frac{\Gamma; \Delta^{\vec{b}} \Longrightarrow B \setminus c}{\Gamma; \Delta^{\vec{b}} \Longrightarrow A \oplus B \setminus c}\,\oplus\mathrm{R}_2$$

$$\frac{\Gamma; \Delta^{\vec{b}}, A^{\top} \Longrightarrow C \setminus c' \qquad \Gamma; \Delta^{\vec{b}}, B^{\top} \Longrightarrow C \setminus c''}{\Gamma; \Delta^{\vec{b}}, (A \oplus B)^d \Longrightarrow C \setminus d \doteq \top \wedge c' \wedge c''}\,\oplus\mathrm{L}$$

$$\textit{No } \mathbf{0} \textit{ right rule} \quad \frac{}{\Gamma; \Delta^{\vec{b}}, (\mathbf{0})^d \Longrightarrow C \setminus d \doteq \top}\,\mathbf{0}\mathrm{L}$$

Quantifiers. The interaction of the quantifiers with resource management is benign, and limited requiring the principal propositions of left rules to occur.

$$\frac{\Gamma; \Delta^{\vec{b}} \Longrightarrow [a/x]A \setminus c}{\Gamma; \Delta^{\vec{b}} \Longrightarrow \forall x.\ A \setminus c}\,\forall\mathrm{R}^a \qquad \frac{\Gamma; \Delta^{\vec{b}}, ([t/x]A)^{\top} \Longrightarrow C \setminus c}{\Gamma; \Delta^{\vec{b}}, (\forall x.\ A)^d \Longrightarrow C \setminus d \doteq \top \wedge c}\,\forall\mathrm{L}$$

$$\frac{\Gamma; \Delta^{\vec{b}} \Longrightarrow [t/x]A \setminus c}{\Gamma; \Delta^{\vec{b}} \Longrightarrow \exists x.\ A \setminus c}\,\exists\mathrm{R} \qquad \frac{\Gamma; \Delta^{\vec{b}}, ([a/x]A)^{\top} \Longrightarrow C \setminus c}{\Gamma; \Delta^{\vec{b}}, (\exists x.\ A)^d \Longrightarrow C \setminus d \doteq \top \wedge c}\,\exists\mathrm{L}^a$$

Exponentials. There are two natural formulations of the $\supset$L and !R rules: we either do not pass any linear hypotheses to the relevant premiss as shown below, or we pass all linear hypotheses but constrain them not to be used. Depending on the design of the implementation, one or the other might be preferable.

$$\frac{(\Gamma, A); \Delta^{\vec{b}} \Longrightarrow B \setminus c}{\Gamma; \Delta^{\vec{b}} \Longrightarrow A \supset B \setminus c} \supset\mathrm{R} \qquad \frac{\Gamma; \cdot \Longrightarrow A \setminus c' \qquad \Gamma; \Delta^{\vec{b}}, B^{\top} \Longrightarrow C \setminus c''}{\Gamma; \Delta^{\vec{b}}, (A \supset B)^{d} \Longrightarrow C \setminus d \doteq \top \wedge c' \wedge c''} \supset\mathrm{L}$$

$$\frac{\Gamma; \cdot \Longrightarrow A \setminus c}{\Gamma; \Delta^{\vec{b}} \Longrightarrow !A \setminus \vec{b} \doteq \bot \wedge c} !\mathrm{R} \qquad \frac{(\Gamma, A); \Delta^{\vec{b}} \Longrightarrow C \setminus c}{\Gamma; \Delta^{\vec{b}}, (!A)^{d} \Longrightarrow C \setminus d \doteq \top \wedge c} !\mathrm{L}$$

We write Θ for an assignment of $\top$ or $\bot$ to all occurrence variables. Applying such an assignment to an annotated context of linear hypotheses is defined as

$$\begin{array}{rcll} [\Theta]\cdot & = & \cdot, & \\ [\Theta](\Delta, w^{b}{:}A) & = & [\Theta]\Delta, w{:}A & \text{if } [\Theta]b = \top, \text{ and} \\ [\Theta](\Delta, w^{b}{:}A) & = & [\Theta]\Delta & \text{if } [\Theta]b = \bot. \end{array}$$

Applying an assignment of a derivation simply applies it to every sequent in the derivation and erases the occurrence constraints.

We should then have the following soundness and completeness theorems.[1]

Theorem 3.4 (Soundness of Occurrence Constraints) *If* $\mathcal{D} :: (\Gamma; \Delta^{\vec{b}} \Longrightarrow C \setminus c)$ *and* $\models [\Theta]c$ *then* $[\Theta]\mathcal{D} :: (\Gamma; [\Theta]\Delta^{\vec{b}} \Longrightarrow C)$.

Theorem 3.5 (Completeness of Occurrence Constraints) *If* $\mathcal{D} :: (\Gamma; \Delta \Longrightarrow C)$ *then* $\mathcal{D}' :: (\Gamma; \Delta^{\vec{\top}} \Longrightarrow C \setminus c)$ *and there is an assignment* Θ *such that* $\models [\Theta]c$ *and* $\mathcal{D} = [\Theta]\mathcal{D}'$.

One or both of these "theorems" may have to be generalized before they can be proved by induction.

How do we check constraints for satisfiability? We have not fully investigated this issue to date. One possibility (suggested by Harland and Pym [**?**]) is to map them to Boolean constraints, which would make them amenable to standard Boolean constraint solving techniques. It seems, however, that this would lead to an unnecessarily complex procedure. We sketch here a set of of rules which may be used to simplify constraints put them into a normal form which should always have solutions. The rules may be incomplete (and certainly would require additional invariants, as we remark below). The apply to any conjunct of the global constraint c.

[1] [*Warning: at present I have not proven these.*]

$$
\begin{array}{rcl}
\top \doteq \top & \longrightarrow & \mathrm{tt} \\
\bot \doteq \bot & \longrightarrow & \mathrm{tt} \\
\top \doteq \bot & & \text{unsatisfiable} \\
\bot \doteq \top & & \text{unsatisfiable} \\
o \doteq b & \longrightarrow & \mathrm{tt} \quad \text{and substitute } b \text{ for } o \text{ everywhere} \\
\top + \bot \doteq b & \longrightarrow & b \doteq \top \\
\bot + \top \doteq b & \longrightarrow & b \doteq \top \\
\top + \top \doteq b & & \text{unsatisfiable} \\
\bot + \bot \doteq b & \longrightarrow & b \doteq \bot \\
o + \top \doteq b & \longrightarrow & o \doteq \bot \wedge b \doteq \top \\
\top + o \doteq b & \longrightarrow & o \doteq \bot \wedge b \doteq \top \\
o + \bot \doteq b & \longrightarrow & o \doteq b \\
\bot + o \doteq b & \longrightarrow & o \doteq b \\
o_1 + o_2 \doteq \bot & \longrightarrow & o_1 \doteq \bot \wedge o_2 \doteq \bot \\
o_1 + o_2 \doteq \top & & \text{in normal form} \\
o_1 + o_2 \doteq o_3 & & \text{in normal form}
\end{array}
$$

The last two cases of normal forms do not imply satisfiability. For example, $o + o \doteq \top$ is not satisfiable. Similarly, $o + o' \doteq o'$ entails that $o \doteq \bot$, which might be inconsistent with other constraints. However, I believe that there is a natural ordering on occurrence variables (and an induced ordering among atomic constraints) which can guarantee that certain cases of this form can not arise, or arise only in a limited number of circumstances which can be checked easily.

> [**Extra Credit Assignment:** *Complete the rules above as needed to guarantee the satisfiability of normal forms for equations which arise from proof search and constraint simplification and proof them correct.*]