Chapter 4

Linear λ-Calculus

In intuitionistic logic, proofs are related to functional programs via the *Curry-Howard isomorphism* [CF58, How69]. Howard observed that there is a bijective correspondence between proofs in intuitionistic propositional natural deduction and simply-typed λ-terms. A related observation on proof in combinatory logic had been made previously by Curry.

A generalization of this observation to include quantifiers later gives rise to the rich field of type theory, which we will analyze in Chapter **??**. Here we study the basic correspondence, extended to the case of linear logic.

A linear λ-calculus of proof terms will be useful for us in various circumstances. First of all, it gives a compact and faithful representation of proofs as terms. Proof checking is reduced to type-checking in a λ-calculus. For example, if we do not trust the implementation of our theorem prover, we can instrument it to generate proof terms which can be verified independently. Secondly, the terms in the λ-calculus provide the core of a functional language with an expressive type system, in which statements such as "*this function will use its argument exactly once*" can be formally expressed and checked. Thirdly, linear λ-terms can serve as an expressive representation language within a *logical framework*, a general meta-language for the formalization of deductive systems.

4.1 Proof Terms

We now assign proof terms to the system of linear natural deduction. Our main criterion for the design of the proof term language is that the proof terms should reflect the structure of the deduction as closely as possible. Moreover, we would like every valid proof term to uniquely determine a natural deduction. Because of the presence of $\top$, this strong property will fail, but a slightly weaker and, from the practical point of view, sufficient property holds. Under the Curry-Howard isomorphism, a proposition corresponds to a type in the proof term calculus. We will there call a proof term *well-typed* if it represents a deduction.

The proof term assignment is defined via the judgment $\Gamma; \Delta \vdash M : A$, where each formula in Γ and Δ is labelled. We also use $M \longrightarrow_\beta M'$ for the local reduction and $M : A \longrightarrow_\eta M'$ for the local expansion, both expressed on proof terms. The type on the left-hand side of the expansion reminds is a reminder that this rule only applies to term of the given type (contexts are elided here).

Hypotheses. We use the label of the hypotheses as the name for a variable in the proof terms. There are no reductions or expansions specific to variables, although variables of non-atomic type may be expanded by the later rules.

$$\frac{}{\Gamma; (\cdot, w{:}A) \vdash w : A}\,w \qquad \frac{}{(\Gamma_1, u{:}A, \Gamma_2); \cdot \vdash u : A}\,u$$

Multiplicative Connectives. Linear implication corresponds to a *linear function types* with corresponding linear abstraction and application. We distinguish them from unrestricted abstraction and application by a "hat". In certain circumstances, this may be unnecessary, but here we want to reflect the proof structure as directly as possible.

$$\frac{\Gamma; (\Delta, w{:}A) \vdash M : B}{\Gamma; \Delta \vdash \hat{\lambda} w{:}A.\ M : A \multimap B} \multimap\mathrm{I}^w$$

$$\frac{\Gamma; \Delta \vdash M : A \multimap B \qquad \Gamma; \Delta' \vdash N : A}{\Gamma; (\Delta \times \Delta') \vdash M\,\hat{}\,N : B} \multimap\mathrm{E}$$

$$\begin{array}{rcl} (\hat{\lambda} w{:}A.\ M)\,\hat{}\,N & \longrightarrow_\beta & [N/w]M \\ M : A \multimap B & \longrightarrow_\eta & \hat{\lambda} w{:}A.\ M\,\hat{}\,w \end{array}$$

In the rules for the simultaneous conjunction, the proof term for the elimination inference is a **let** form which deconstructs a pair, naming the components. The linearity of the two new hypotheses means that the variables must both be used in M.

$$\frac{\Gamma; \Delta_1 \vdash M : A \qquad \Gamma; \Delta_2 \vdash N : B}{\Gamma; (\Delta_1 \times \Delta_2) \vdash M \otimes N : A \otimes B} \otimes\mathrm{I}$$

$$\frac{\Gamma; \Delta \vdash M : A \otimes B \quad \Gamma; (\Delta', w_1{:}A, w_2{:}B) \vdash N : C}{\Gamma; (\Delta' \times \Delta) \vdash \mathbf{let}\ w_1 \otimes w_2 = M\ \mathbf{in}\ N : C} \otimes\mathrm{E}^{w_1, w_2}$$

The reduction and expansion mirror the local reduction and expansion for deduction as the level of proof terms. We do not reiterate them here, but simply give the proof term reduction.

$$\begin{array}{rcl} \mathbf{let}\ w_1 \otimes w_2 = M_1 \otimes M_2\ \mathbf{in}\ N & \longrightarrow_\beta & [M_1/w_1, M_2/w_2]N \\ M : A \otimes B & \longrightarrow_\eta & \mathbf{let}\ w_1 \otimes w_2 = M\ \mathbf{in}\ w_1 \otimes w_2 \end{array}$$

The unit type allows us to consume linear hypotheses without introducing new linear ones.

$$\frac{}{\Gamma; \cdot \vdash \star : \mathbf{1}}\,\mathbf{1}\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash M : \mathbf{1} \qquad \Gamma; \Delta' \vdash N : C}{\Gamma; (\Delta' \times \Delta) \vdash \mathbf{let}\ \star = M\ \mathbf{in}\ N : C}\,\mathbf{1}\mathrm{E}$$

$$\begin{array}{rcl} \mathbf{let}\ \star = M\ \mathbf{in}\ N & \longrightarrow_\beta & N \\ M : \star & \longrightarrow_\eta & \mathbf{let}\ \star = M\ \mathbf{in}\ \star \end{array}$$

Additive Connectives. As we have seen from the embedding of intuitionistic in linear logic, the simultaneous conjunction represents products from the simply-typed λ-calculus.

$$\frac{\Gamma; \Delta \vdash M : A \qquad \Gamma; \Delta \vdash N : B}{\Gamma; \Delta \vdash \langle M, N \rangle : A \& B}\,\&\mathrm{I}$$

$$\frac{\Gamma; \Delta \vdash M : A \& B}{\Gamma; \Delta \vdash \mathrm{fst}\, M : A}\,\&\mathrm{E_L} \qquad \frac{\Gamma; \Delta \vdash M : A \& B}{\Gamma; \Delta \vdash \mathrm{snd}\, M : B}\,\&\mathrm{E_R}$$

The local reduction are also the familiar ones.

$$\begin{array}{rcl} \mathrm{fst}\,\langle M_1, M_2 \rangle & \longrightarrow_\beta & M_1 \\ \mathrm{snd}\,\langle M_1, M_2 \rangle & \longrightarrow_\beta & M_2 \\ M : A \& B & \longrightarrow_\eta & \langle \mathrm{fst}\, M, \mathrm{snd}\, M \rangle \end{array}$$

The additive unit corresponds to a unit type with no operations on it.

$$\frac{}{\Gamma; \Delta \vdash \langle\,\rangle : \top}\,\top\mathrm{I} \qquad \textit{No } \top \textit{ elimination}$$

The additive unit has no elimination and therefore no reduction. However, it still admits an expansion, which witnesses the local completeness of the rules.

$$M : \top \quad \longrightarrow_\eta \quad \langle\,\rangle$$

The disjunction (or *disjoint sum* when viewed as a type) uses injection and case as constructor and destructor forms, respectively. We annotated the injections with a type to preserve the property that any well-typed term has a unique type.

$$\frac{\Gamma; \Delta \vdash M : A}{\Gamma; \Delta \vdash \mathrm{inl}^B : A \oplus B}\,\oplus\mathrm{I_L} \qquad \frac{\Gamma; \Delta \vdash M : B}{\Gamma; \Delta \vdash \mathrm{inr}^A : A \oplus B}\,\oplus\mathrm{I_R}$$

$$\frac{\Gamma; \Delta \vdash M : A \oplus B \quad \Gamma; (\Delta', w_1{:}A) \vdash N_1 : C \quad \Gamma; (\Delta', w_2{:}B) \vdash N_2 : C}{\Gamma; (\Delta' \times \Delta) \vdash \mathbf{case}\ M\ \mathbf{of}\ \mathrm{inl}\, w_1 \Rightarrow N_1 \mid \mathrm{inr}\, w_2 \Rightarrow N_2 : C}\,\oplus\mathrm{E}^{w_1, w_2}$$

The reductions are just like the ones for disjoint sums in the simply-typed λ-caclulus.

$$\begin{array}{lcl} \mathbf{case}\ \mathrm{inl}^B\, M\ \mathbf{of}\ \mathrm{inl}\, w_1 \Rightarrow N_1 \mid \mathrm{inr}\, w_2 \Rightarrow N_2 & \longrightarrow_\beta & [M/w_1]N_1 \\ \mathbf{case}\ \mathrm{inr}^A\, M\ \mathbf{of}\ \mathrm{inl}\, w_1 \Rightarrow N_1 \mid \mathrm{inr}\, w_2 \Rightarrow N_2 & \longrightarrow_\beta & [M/w_2]N_2 \end{array}$$

$$M : A \oplus B \quad \longrightarrow_\eta \quad \mathbf{case}\ M\ \mathbf{of}\ \mathrm{inl}\, w_1 \Rightarrow \mathrm{inl}^B\, w_1 \mid \mathrm{inr}\, w_2 \Rightarrow \mathrm{inr}^A\, w_2$$

For the additive falsehood, there is no introduction rule. It corresponds to a *void type* without any values. Consequently, there is no reduction. Once again we annotate the abort constructor in order to guarantee uniqueness of types.

$$\textit{No } \mathbf{0} \textit{ introduction} \qquad \frac{\Gamma; \Delta \vdash M : \mathbf{0}}{\Gamma; (\Delta' \times \Delta) \vdash \mathrm{abort}^C\, M : C}\,\mathbf{0}\mathrm{E}$$

$$M : \mathbf{0} \quad \longrightarrow_\eta \quad \mathrm{abort}^{\mathbf{0}}\, M$$

Exponentials. Unrestricted implication corresponds to the usual *function type* from the simply-typed λ-calculus. For consistency, we will still write $A \supset B$ instead of $A \to B$, which is more common in λ-calculus. Note that the argument of an unrestricted application may not mention any linear variables.

$$\frac{(\Gamma, u{:}A); \Delta \vdash M : B}{\Gamma; \Delta \vdash \lambda u{:}A.\ M : A \supset B}\,{\supset}\mathrm{I}^u$$

$$\frac{\Gamma; \Delta \vdash M : A \supset B \qquad \Gamma; \cdot \vdash N : A}{\Gamma; \Delta \vdash M\, N : B}\,{\supset}\mathrm{E}$$

The reduction and expansion are the origin of the β and η rules names due to Church [**?**].

$$\begin{array}{rcl} (\lambda u{:}A.\ M)\, N & \longrightarrow_\beta & [N/u]M \\ M : A \supset B & \longrightarrow_\eta & \lambda u{:}A.\ M\, u \end{array}$$

The rules for the *of course* operator allow us to name term of type $!A$ and use it freely in further computation.

$$\frac{\Gamma; \cdot \vdash M : A}{\Gamma; \cdot \vdash\, !M : !A}\,!\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash M : !A \qquad (\Gamma, u{:}A); \Delta' \vdash N : C}{\Gamma; (\Delta' \times \Delta) \vdash \mathbf{let}\ !u = M\ \mathbf{in}\ N : C}\,!\mathrm{E}^u$$

$$\begin{array}{rcl} \mathbf{let}\ !u = !M\ \mathbf{in}\ N & \longrightarrow_\beta & [M/u]N \\ M : !A & \longrightarrow_\eta & \mathbf{let}\ !u = M\ \mathbf{in}\ !u \end{array}$$

Below is a summary of the linear λ-calculus with the β-reduction and η-expansion rules.

$$
\begin{array}{lcll}
M & ::= & w & \textit{Linear Variables} \\
 & & \mid \hat{\lambda} w{:}A.\ M \mid M_1 \hat{}\ M_2 & A \multimap B \\
 & & \mid M_1 \otimes M_2 \mid \mathbf{let}\ w_1 \otimes w_2 = M\ \mathbf{in}\ M' & A \otimes B \\
 & & \mid \star \mid \mathbf{let}\ \star = M\ \mathbf{in}\ M' & \mathbf{1} \\
 & & \mid \langle M_1, M_2 \rangle \mid \mathrm{fst}\, M_1 \mid \mathrm{snd}\, M_2 & A \& B \\
 & & \mid \langle\, \rangle & \top \\
 & & \mid \mathrm{inl}^B\, M \mid \mathrm{inr}^A\, M & A \oplus B \\
 & & \quad \mid (\,\mathbf{case}\, M\ \mathbf{of}\ \mathrm{inl}\, w_1 \Rightarrow M_1 \mid \mathrm{inr}\, w_2 \Rightarrow M_2) & \\
 & & \mid \mathrm{abort}^C\, M & \mathbf{0} \\
 & & \mid u & \textit{Unrestricted Variables} \\
 & & \mid \lambda u{:}A.\ M \mid M_1\, M_2 & A \supset B \\
 & & \mid\, !M \mid \mathbf{let}\ u = M\ \mathbf{in}\ M' & !A
\end{array}
$$

Below is a summary of the β-reduction rules, which correspond to local reductions of natural deductions.

$$
\begin{array}{rcll}
(\hat{\lambda} w{:}A.\ M) \hat{}\ N & \longrightarrow_\beta & [N/w]M & A \multimap B \\
\mathbf{let}\ w_1 \otimes w_2 = M_1 \otimes M_2\ \mathbf{in}\ N & \longrightarrow_\beta & [M_1/w_1, M_2/w_2]N & A \otimes B \\
\mathbf{let}\ \star = M\ \mathbf{in}\ N & \longrightarrow_\beta & N & \mathbf{1} \\
\mathrm{fst}\, \langle M_1, M_2 \rangle & \longrightarrow_\beta & M_1 & A \& B \\
\mathrm{snd}\, \langle M_1, M_2 \rangle & \longrightarrow_\beta & M_2 & \\
\textit{No } \top \textit{ reduction} & & & \\
\mathbf{case}\ \mathrm{inl}^B\, M\ \mathbf{of}\ \mathrm{inl}\, w_1 \Rightarrow N_1 \mid \mathrm{inr}\, w_2 \Rightarrow N_2 & \longrightarrow_\beta & [M/w_1]N_1 & A \oplus B \\
\mathbf{case}\ \mathrm{inr}^A\, M\ \mathbf{of}\ \mathrm{inl}\, w_1 \Rightarrow N_1 \mid \mathrm{inr}\, w_2 \Rightarrow N_2 & \longrightarrow_\beta & [M/w_1]N_2 & \\
\textit{No } \mathbf{0} \textit{ reduction} & & & \\
(\lambda u{:}A.\ M)\, N & \longrightarrow_\beta & [N/u]M & A \supset B \\
\mathbf{let}\ !u = !M\ \mathbf{in}\ N & \longrightarrow_\beta & [M/u]N & !A
\end{array}
$$

Next is a summary of the η-expansion rules, which correspond to local expansions of natural deductions.

$$
\begin{array}{rcl}
M : A \multimap B & \longrightarrow_\eta & \hat{\lambda} w{:}A.\ M \hat{}\ w \\
M : A \otimes B & \longrightarrow_\eta & \mathbf{let}\ w_1 \otimes w_2 = M\ \mathbf{in}\ w_1 \otimes w_2 \\
M : \star & \longrightarrow_\eta & \mathbf{let}\ \star = M\ \mathbf{in}\ \star \\
M : A \& B & \longrightarrow_\eta & \langle \mathrm{fst}\, M, \mathrm{snd}\, M \rangle \\
M : \top & \longrightarrow_\eta & \langle\, \rangle \\
M : A \oplus B & \longrightarrow_\eta & \mathbf{case}\ M\ \mathbf{of}\ \mathrm{inl}\, w_1 \Rightarrow \mathrm{inl}^B\, w_1 \mid \mathrm{inr}\, w_2 \Rightarrow \mathrm{inr}^A\, w_2 \\
M : \mathbf{0} & \longrightarrow_\eta & \mathrm{abort}^{\mathbf{0}}\, M \\
M : A \supset B & \longrightarrow_\eta & \lambda u{:}A.\ M\, u \\
M : !A & \longrightarrow_\eta & \mathbf{let}\ !u = M\ \mathbf{in}\ !u
\end{array}
$$

We have the following fundamental properties. Uniqueness, where claimed, holds only up to renaming of bound variables.

Theorem 4.1 (Properties of Proof Terms)

1. *If* $\Gamma; \Delta \vdash A$ *then* $\Gamma; \Delta \vdash M : A$ *for a unique* M.
2. *If* $\Gamma; \Delta \vdash M : A$ *then* $\Gamma; \Delta \vdash A$.

Proof: By straightforward inductions over the given derivations. $\square$

Types are also unique for well-typed terms (see Exercise 4.1). Uniqueness of derivations fails, that is, a proof term does not uniquely determine its derivation, even under identical contexts. A simple counterexample is provided by the following two derivations (with the empty unrestricted context elided).

$$\dfrac{\dfrac{}{w{:}\top \vdash \langle\,\rangle : \top}\top I \qquad \dfrac{}{\cdot \vdash \langle\,\rangle : \top}\top I}{w{:}\top \vdash \langle\,\rangle \otimes \langle\,\rangle : \top \otimes \top}\otimes I \qquad \dfrac{\dfrac{}{\cdot \vdash \langle\,\rangle : \top}\top I \qquad \dfrac{}{w{:}\top \vdash \langle\,\rangle : \top}\top I}{w{:}\top \vdash \langle\,\rangle \otimes \langle\,\rangle : \top \otimes \top}\otimes I$$

It can be shown that linear hypotheses which are absorbed by $\top I$ are the only source of only ambiguity in the derivation. A similar ambiguity already exists in the sense that any proof term remains valid under weakening in the intuitionistic context: whenever $\Gamma; \Delta \vdash M : A$ then $(\Gamma, \Gamma'); \Delta \vdash M : A$. So this phenomenon is not new to the linear λ-calculus, and is in fact a useful identification of derivations which differ in "irrelevant" details, that is, unused or absorbed hypotheses.

4.2 Exercises

Exercise 4.1 Prove that if $\Gamma; \Delta \vdash M : A$ and $\Gamma; \Delta \vdash M : A'$ then $A = A'$.