Linear Logic

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Chapter 1

Natural Deduction

> Ich wollte zunächst einmal einen Formalismus aufstellen, der dem wirklichen Schließen möglichst nahe kommt. So ergab sich ein „Kalkül des natürlichen Schließens“.
>
> — Gerhard Gentzen
> *Untersuchungen über das logische Schließen* [Gen35]

In this chapter we explore ways to define logics, or, which comes to the same thing, ways to give meaning to logical connectives. Our fundamental notion is that of a *judgment* based on *evidence*. For example, we might make the judgment "*It is raining*" based on visual evidence. Or we might make the judgment "*'A implies A' is true for any proposition A*" based on a derivation. The use of the notion of a judgment as conceptual prior to the notion of proposition has been advocated by Martin-Löf [ML85a, ML85b]. Certain forms of judgments frequently recur and have therefore been investigated in their own right, prior to logical considerations. Two that we will use are *hypothetical judgments* and *parametric jugments* (the latter is sometimes called *general judgment* or *schematic judgment*).

A hypothetical judgment has the form "J_2 *under hypothesis* J_1". We consider this judgment evident if we are prepared to make the judgment J_2 once provided with evidence for J_1. Formal evidence for a hypothetical judgment is a *hypothetical derivation* where we can freely use the hypothesis J_1 in the derivation of J_2. Note that hypotheses need not be used, and could be used more than once. In contrast, derivations of *linear hypothetical judgments* introduced later are restricted so the the linear hypotheses must be used exactly once.

A parametric judgment has the form "J *for any* a" where a is a *parameter* which may occur in J. We make this judgment if we are prepared to make the judgment $[O/a]J$ for arbitrary objects O of the right category. Here $[O/a]J$ is our notation for substituting the object O for parameter a in the judgment J. Formal evidence for a parametric judgment J is a *parametric derivation* with free occurrences of the parameter a.

Formal evidence for a judgment in form of a derivation is usually written in two-dimensional notation:

$$\begin{array}{c} \mathcal{D} \\ J \end{array}$$

if $\mathcal{D}$ is a derivation of J. For the sake of brevity we sometimes use the alternative notation $\mathcal{D} of J$. A hypothetical judgment is written as

$$\begin{array}{c} \overline{J_1}\, u \\ \vdots \\ J_2 \end{array}$$

where u is a label which identifies the hypothesis J_1. We use the labels to guarantee that hypotheses which are introduced during the reasoning process are not used outside their scope.

The separation of the notion of judgment and proposition and the corresponding separation of the notion of evidence and proof sheds new light on various styles that have been used to define logical systems.

An axiomatization in the style of Hilbert [Hil92], for example, arises when one defines a judgment "*A is true*" without the use of hypothetical judgments. Such a definition is highly economical in its use of judgments, which has to be compensated by a liberal use of implication in the axioms. When we make proof structure explicit in such an axiomatization, we arrive at combinatory logic [Cur30].

A categorical logic [LS86] arises when the basic judgment is not truth, but entailment "*A entails B*". Once again, presentations are highly economical and do not need to seek recourse in complex judgment forms (at least for the propositional fragment). But derivations often require many hypotheses, which means that we need to lean rather heavily on conjunction here. Proofs are realized by morphisms which are an integral part of the machinery of category theory.

While these are interesting and in many ways useful approaches to logic specification, neither of them comes particularly close to capturing the practice of mathematical reasoning. This was Gentzen's point of departure for the design of a system of *natural deduction* [Gen35]. From our point of view, this system is based on the simple judgment "*A is true*", but relies critically on hypothetical and parametric judgments. In addition to being extremely elegant, it has the great advantage that one can define all logical connectives without reference to any other connective. This principle of modularity extends to the meta-theoretic study of natural deduction and simplifies considering fragments and extension of logics. Since we will consider many fragments and extension, this *orthogonality* of the logical connectives is a critical consideration. There is another advantage to natural deduction, namely that its proofs are isomorphic to the terms in a λ-calculus via the so-called Curry-Howard isomorphism [How69], which establishes many connections to functional programming.

Finally, we arrive at the *sequent calculus* (also introduced by Gentzen in his seminal paper [Gen35]) when we split the single judgment of truth into two:

"*A is an assumption*" and "*A is a true conclusion*". While we still employ the machinery of parametric and hypothetical judgments, we now need an explicit rule to state that "*A is an assumption*" is sufficient evidence for "*A is a true conclusion*". The reverse, namely that if "*A is a true conclusion*" then "*A may be used as an assumption*" is the Cut rule which he proved to be redundant in his *Hauptsatz*. For Gentzen the sequent calculus was primarily a technical device to prove consistency of his system of natural deduction, but it exposes many details of the fine structure of proofs in such a clear manner, that most presentations of linear and related logics employ sequent calculus. The laws governing the structure of proofs, however, are more complicated than the Curry-Howard isomorphism for natural deduction might suggest and are still the subject of study [Her95, Pfe95].

We choose natural deduction as our definitional formalism as the purest and most widely applicable. Later we justify the sequent calculus as a calculus of proof search for natural deduction and explicitly relate the two forms of presentation.

We begin by introducing natural deduction for intuitionistic logic, exhibiting its basic principles. We then enrich it to capture linear connectives.

1.1 Intuitionistic Natural Deduction

The system of natural deduction we describe below is basically Gentzen's system NJ [Gen35] or the system which may be found in Prawitz [Pra65]. The calculus of natural deduction was devised by Gentzen in the 1930's out of a dissatisfaction with axiomatic systems in the Hilbert tradition, which did not seem to capture mathematical reasoning practices very directly. Instead of a number of axioms and a small set of inference rules, valid deductions are described through inference rules only, which at the same time explain the meaning of the logical quantifiers and connectives in terms of their proof rules.

A language of (first-order) *terms* is built up from *variables* x, y, *etc.*, *function symbols* f, g, *etc.*, each with a unique arity, and *parameters* a, b, *etc.* in the usual way.

$$\textit{Terms} \quad t \quad ::= \quad x \mid a \mid f(t_1, \ldots, t_n)$$

A constant c is simply a function symbol with arity 0 and we write c instead of $c()$. Exactly which function symbols are available is left unspecified in the general development of predicate logic and only made concrete for specific theories, such as the theory of natural numbers. However, variables and parameters are always available. We will use t and s to range over terms.

The language of *propositions* is built up from *predicate symbols* P, Q, *etc.* and terms in the usual way.

$$\begin{array}{lcl} \textit{Propositions} \quad A & ::= & P(t_1, \ldots, t_n) \mid A_1 \wedge A_2 \mid A_1 \supset A_2 \mid A_1 \vee A_2 \mid \neg A \\ & & \mid \bot \mid \top \mid \forall x.\, A \mid \exists x.\, A \end{array}$$

A propositional constant P is simply a predicate symbol with no arguments and we write P instead of $P()$. We will use A, B, and C to range over propositions.

Exactly which predicate symbols are available is left unspecified in the general development of predicate logic and only made concrete for specific theories.

The notions of *free* and *bound* variables in terms and propositions are defined in the usual way: the variable x is bound in propositions of the form $\forall x.\, A$ and $\exists x.\, A$. We use parentheses to disambiguate and assume that $\wedge$ and $\vee$ bind more tightly than $\supset$. It is convenient to assume that propositions have no free individual variables; we use parameters instead where necessary. Our notation for substitution is $[t/x]A$ for the result of substituting the term t for the variable x in A. Because of the restriction on occurrences of free variables, we can assume that t is free of individual variables, and thus capturing cannot occur.

The main judgment of natural deduction is "C *is true*" written as $\vdash C$, from hypotheses $\vdash A_1, \ldots, \vdash A_n$. We will model this as a hypothetical judgment. This means that certain structural properties of derivations are tacitly assumed, independently of any logical inferences. In essence, these assumptions explain what hypothetical judgments are.

Hypothesis. If we have a hypothesis $\vdash A$ than we can conclude $\vdash A$.

Weakening. Hypotheses need not be used.

Duplication. Hypotheses can be used more than once.

Exchange. The order in which hypotheses are introduced is irrelevant.

In keeping with general mathematical practice in the discussion of natural deduction, we might sometimes omit the turnstile $\vdash$ and let a proposition A itself stand for the judgment $\vdash A$. It is important to keep in mind that this is merely a shorthand, and that we are defining a judgment via inference rules in the same manner as in earlier chapters in this book.

In natural deduction each logical connective and quantifier is characterized by its *introduction rule(s)* which specifies how to infer that a conjunction, disjunction, *etc.* is true. The *elimination rule* for the logical constant tells what other truths we can deduce from the truth of a conjunction, disjunction, *etc.* Introduction and elimination rules must match in a certain way in order to guarantee that the rules are meaningful and the overall system can be seen as capturing mathematical reasoning.

The first is a *local soundness* property: if we introduce a connective and then immediately eliminate it, we should be able to erase this detour and find a more direct derivation of the conclusion without using the connective. If this property fails, the elimination rules are too strong: they allow us to conclude more than we should be able to know.

The second is a *local completeness* property: we can eliminate a connective in a way which retains sufficient information to reconstitute it by an introduction rule. If this property fails, the elimination rules are too weak: the do not allow us to conclude everything we should be able to know.

We provide evidence for local soundness and completeness of the rules by means of *local reduction* and *expansion* judgments, which relate proofs of the same proposition.

One of the important principles of natural deduction is that each connective should be defined only in terms of inference rules without reference to other logical connectives or quantifiers. We refer to this as *orthogonality* of the connectives. It means that we can understand a logical system as a whole by understanding each connective separately. It also allows us to consider fragments and extensions directly and it means that the investigation of properties of a logical system can be conducted in a modular way.

We now show the introduction and elimination rules, local reductions and expansion for each of the logical connectives in turn. The rules are summarized on page 1.1.

Conjunction. $A \wedge B$ should be true if both A and B are true. Thus we have the following introduction rule.

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\,\wedge\mathrm{I}$$

If we consider this as a complete definition, we should be able to recover both A and B if we know $A \wedge B$. We are thus led to two elimination rules.

$$\frac{\vdash A \wedge B}{\vdash A}\,\wedge\mathrm{E_L} \qquad \frac{\vdash A \wedge B}{\vdash B}\,\wedge\mathrm{E_R}$$

To check our intuition we consider a deduction which ends in an introduction followed by an elimination:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \vdash B\end{array}}{\vdash A \wedge B}\,\wedge\mathrm{I}}{\vdash A}\,\wedge\mathrm{E_L}$$

Clearly, it is unnecessary to first introduce the conjunction and then eliminate it: a more direct proof of the same conclusion from the same (or fewer) assumptions would be simply

$$\begin{array}{c}\mathcal{D}\\ \vdash A\end{array}$$

Formulated as a transformation or *reduction* between derivations we have

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \vdash B\end{array}}{\vdash A \wedge B}\,\wedge\mathrm{I}}{\vdash A}\,\wedge\mathrm{E_L} \Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ \vdash A\end{array}$$

and symmetrically

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ \vdash A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \vdash B\end{array}}{\vdash A \wedge B}\,\wedge\mathrm{I}}{\vdash B}\,\wedge\mathrm{E_R} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{E}\\ \vdash B\end{array}$$

The new judgment

$$\begin{array}{c}\mathcal{D}\\ \vdash A\end{array} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{E}\\ \vdash A\end{array}$$

relates derivations with the same conclusion. We say $\mathcal{D}$ *locally reduces to* $\mathcal{E}$. Since local reductions are possible for both elimination rules for conjunction, our rules are locally sound. To show that the rules are locally complete we show how to reintroduce a conjunction from its components in the form of a local expansion.

$$\begin{array}{c}\mathcal{D}\\ \vdash A \wedge B\end{array} \quad \Longrightarrow_E \quad \cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ \vdash A \wedge B\end{array}}{\vdash A}\,\wedge\mathrm{E_L} \qquad \cfrac{\begin{array}{c}\mathcal{D}\\ \vdash A \wedge B\end{array}}{\vdash B}\,\wedge\mathrm{E_R}}{\vdash A \wedge B}\,\wedge\mathrm{I}$$

Implication. To derive $\vdash A \supset B$ we assume $\vdash A$ and then derive $\vdash B$. Written as a hypothetical judgment:

$$\cfrac{\begin{array}{c}\overline{\vdash A}\;u\\ \vdots\\ \vdash B\end{array}}{\vdash A \supset B}\,\supset\mathrm{I}^u$$

We must be careful that the hypothesis $\vdash A$ is available only in the derivation above the premiss. We therefore label the inference with the name of the hypothesis u, which must not be used already as the name for a hypothesis in the derivation of the premiss. We say that the hypothesis $\vdash A$ labelled u is *discharged* at the inference labelled $\supset\mathrm{I}^u$. A derivation of $\vdash A \supset B$ describes a construction by which we can transform a derivation of $\vdash A$ into a derivation of $\vdash B$: we substitute the derivation of $\vdash A$ wherever we used the assumption $\vdash A$ in the hypothetical derivation of $\vdash B$. The elimination rule expresses this: if we have a derivation of $\vdash A \supset B$ and also a derivation of $\vdash A$, then we can obtain a derivation of $\vdash B$.

$$\frac{\vdash A \supset B \qquad \vdash A}{\vdash B}\,\supset\mathrm{E}$$

The local reduction rule carries out the substitution of derivations explained above.

$$\dfrac{\dfrac{\begin{array}{c}\overline{\vdash A}\,u\\ \mathcal{D}\\ \vdash B\end{array}}{\vdash A \supset B}\supset\mathrm{I}^u \qquad \begin{array}{c}\mathcal{E}\\ \vdash A\end{array}}{\vdash B}\supset\mathrm{E} \qquad \Longrightarrow_R \qquad \begin{array}{c}\mathcal{E}\\ \overline{\vdash A}\,u\\ \mathcal{D}\\ \vdash B\end{array}$$

The final derivation depends on all the hypotheses of $\mathcal{E}$ and $\mathcal{D}$ except u, for which we have substituted $\mathcal{E}$. An alternative notation for this substitution of derivations for hypotheses is $[\mathcal{E}/u]\mathcal{D} :: \vdash B$. The local reduction described above may significantly increase the overall size of the derivation, since the deduction $\mathcal{E}$ is substituted for each occurrence of the assumption labeled u in $\mathcal{D}$ and may thus be replicated many times. The local expansion simply rebuilds the implication.

$$\begin{array}{c}\mathcal{D}\\ \vdash A \supset B\end{array} \qquad \Longrightarrow_E \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash A \supset B\end{array} \qquad \overline{\vdash A}\,u}{\vdash B}\supset\mathrm{E}}{\vdash A \supset B}\supset\mathrm{I}^u$$

Disjunction. $A \vee B$ should be true if either A is true or B is true. Therefore we have two introduction rules.

$$\dfrac{\vdash A}{\vdash A \vee B}\vee\mathrm{I_L} \qquad \dfrac{\vdash B}{\vdash A \vee B}\vee\mathrm{I_R}$$

If we have a hypothesis $\vdash A \vee B$, we do not know how it might be inferred. That is, a proposed elimination rule

$$\dfrac{\vdash A \vee B}{\vdash A}\,?$$

would be incorrect, since a deduction of the form

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{E}\\ \vdash B\end{array}}{\vdash A \vee B}\vee\mathrm{I_R}}{\vdash A}\,?$$

cannot be reduced. As a consequence, the system would be *inconsistent*: if we have at least one theorem (B, in the example) we can prove every formula (A, in the example). How do we use the assumption $A \vee B$ in informal reasoning? We often proceed with a proof by cases: we prove a conclusion C under the assumption A and also show C under the assumption B. We then conclude C, since either A or B by assumption. Thus the elimination rule employs two hypothetical judgments.

$$\dfrac{\vdash A \lor B \qquad \begin{array}{c}\overline{\vdash A}\,u_1\\ \vdots \\ \vdash C\end{array} \qquad \begin{array}{c}\overline{\vdash B}\,u_2\\ \vdots\\ \vdash C\end{array}}{\vdash C}\lor\mathrm{E}^{u_1,u_2}$$

Now one can see that the introduction and elimination rules match up in two reductions. First, the case that the disjunction was inferred by $\lor\mathrm{I_L}$.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash A\end{array}}{\vdash A \lor B}\lor\mathrm{I_L} \qquad \begin{array}{c}\overline{\vdash A}\,u_1\\ \mathcal{E}_1\\ \vdash C\end{array} \qquad \begin{array}{c}\overline{\vdash B}\,u_2\\ \mathcal{E}_2\\ \vdash C\end{array}}{\vdash C}\lor\mathrm{E}^{u_1,u_2} \quad \Longrightarrow_R \quad \begin{array}{c}\dfrac{\mathcal{D}}{\vdash A}\,u_1\\ \mathcal{E}_1\\ \vdash C\end{array}$$

The other reduction is symmetric.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash B\end{array}}{\vdash A \lor B}\lor\mathrm{I_R} \qquad \begin{array}{c}\overline{\vdash A}\,u_1\\ \mathcal{E}_1\\ \vdash C\end{array} \qquad \begin{array}{c}\overline{\vdash B}\,u_2\\ \mathcal{E}_2\\ \vdash C\end{array}}{\vdash C}\lor\mathrm{E}^{u_1,u_2} \quad \Longrightarrow_R \quad \begin{array}{c}\dfrac{\mathcal{D}}{\vdash B}\,u_2\\ \mathcal{E}_2\\ \vdash C\end{array}$$

As in the reduction for implication, the resulting derivation may be longer than the original one. The local expansion is more complicated than for the previous connectives, since we first have to distinguish cases and then reintroduce the disjunction in each branch.

$$\begin{array}{c}\mathcal{D}\\ \vdash A \lor B\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \vdash A \lor B\end{array} \qquad \dfrac{\overline{\vdash A}\,u_1}{\vdash A \lor B}\lor\mathrm{I_L} \qquad \dfrac{\overline{\vdash B}\,u_2}{\vdash A \lor B}\lor\mathrm{I_R}}{\vdash A \lor B}\lor\mathrm{E}^{u_1,u_2}$$

Negation. In order to derive $\neg A$ we assume A and try to derive a contradiction. Thus it seems that negation requires falsehood, and, indeed, in most literature on constructive logic, $\neg A$ is seen as an abbreviation of $A \supset \bot$. In order to give a self-contained explanation of negation by an introduction rule, we employ a judgment that is parametric in a propositional parameter p: If we can derive *any* p from the hypothesis A we conclude $\neg A$.

$$\dfrac{\begin{array}{c}\overline{\vdash A}\,u\\ \vdots\\ \vdash p\end{array}}{\vdash \neg A}\,\neg\mathrm{I}^{p,u} \qquad\qquad \dfrac{\vdash \neg A \qquad \vdash A}{\vdash C}\,\neg\mathrm{E}$$

The elimination rule follows from this view: if we know $\vdash \neg A$ and $\vdash A$ then we can conclude any formula C is true. In the form of a local reduction:

$$\dfrac{\dfrac{\begin{array}{c}\overline{\vdash A}\,u\\ \mathcal{D}\\ \vdash p\end{array}}{\vdash \neg A}\,\neg\mathrm{I}^{p,u} \qquad \begin{array}{c}\mathcal{E}\\ \vdash A\end{array}}{\vdash C}\,\neg\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}\dfrac{\mathcal{E}}{\vdash A}\,u\\ {[C/p]}\mathcal{D}\\ \vdash C\end{array}$$

The substitution $[C/p]\mathcal{D}$ is valid, since $\mathcal{D}$ is parametric in p. The local expansion is similar to the case for implication.

$$\begin{array}{c}\mathcal{D}\\ \vdash \neg A\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \neg A\end{array} \qquad \overline{\vdash A}\,u}{\vdash p}\,\neg\mathrm{E}}{\vdash \neg A}\,\neg\mathrm{I}^{p,u}$$

Truth. There is only an introduction rule for $\top$:

$$\dfrac{}{\vdash \top}\,\top\mathrm{I}$$

Since we put no information into the proof of $\top$, we know nothing new if we have an assumption $\top$ and therefore we have no elimination rule and no local reduction. It may also be helpful to think of $\top$ as a 0-ary conjunction: the introduction rule has 0 premisses instead of 2 and we correspondingly have 0 elimination rules instead of 2. The local expansion allows the replacement of any derivation of $\top$ by $\top$I.

$$\begin{array}{c}\mathcal{D}\\ \vdash \top\end{array} \quad \Longrightarrow_E \quad \dfrac{}{\vdash \top}\,\top\mathrm{I}$$

Falsehood. Since we should not be able to derive falsehood, there is no introduction rule for $\bot$. Therefore, if we can derive falsehood, we can derive everything.

$$\dfrac{\vdash \bot}{\vdash C}\,\bot\mathrm{E}$$

Note that there is no local reduction rule for $\bot$E. It may be helpful to think of $\bot$ as a 0-ary disjunction: we have 0 instead of 2 introduction rules and we correspondingly have to consider 0 cases instead of 2 in the elimination rule. Even though we postulated that falsehood should not be derivable, falsehood could clearly be a consequence of contradictory assumption. For example, $\vdash A \wedge \neg A \supset \bot$ is derivable. While there is no local reduction rule, there still is a local expansion in analogy to the case for disjunction.

$$\begin{array}{c}\mathcal{D}\\ \vdash \bot\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \bot\end{array}}{\vdash \bot}\,\bot\mathrm{E}$$

Universal Quantification. Under which circumstances should $\vdash \forall x.\ A$ be true? This clearly depends on the domain of quantification. For example, if we know that x ranges over the natural numbers, then we can conclude $\forall x.\ A$ if we can prove $[0/x]A$, $[1/x]A$, *etc.* Such a rule is not effective, since it has infinitely many premisses. Thus one usually retreats to rules such as induction. However, in a general treatment of predicate logic we would like to prove statements which are true for *all* domains of quantification. Thus we can only say that $\forall x.\ A$ should be provable if $[a/x]A$ is provable for a new parameter a about which we can make no assumption. Conversely, if we know $\forall x.\ A$, we know that $[t/x]A$ for any term t.

$$\dfrac{\vdash [a/x]A}{\vdash \forall x.\ A}\,\forall\mathrm{I}^a \qquad\qquad \dfrac{\vdash \forall x.\ A}{\vdash [t/x]A}\,\forall\mathrm{E}$$

The label a on the introduction rule is a reminder the parameter a must be "new", that is, it may not occur in any uncancelled assumption in the proof of $[a/x]A$ or in $\forall x.\ A$ itself. In other words, the derivation of the premiss must parametric in a. The local reduction carries out the substitution for the parameter.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash [a/x]A\end{array}}{\vdash \forall x.\ A}\,\forall\mathrm{I}}{\vdash [t/x]A}\,\forall\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}[t/a]\mathcal{D}\\ \vdash [t/x]A\end{array}$$

Here, $[t/a]\mathcal{D}$ is our notation for the result of substituting t for the parameter a throughout the deduction $\mathcal{D}$. For this substitution to preserve the conclusion, we must know that a does not already occur in A. Similarly, we would change the hypotheses if a occurred free in any of the undischarged hypotheses of $\mathcal{D}$. This might render a larger proof incorrect. As an example, consider the formula $\forall x.\ \forall y.\ P(x) \supset P(y)$ which should clearly not be true for all predicates P. The following is *not* a deduction of this formula.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash P(a)}\,u}{\vdash \forall x.\, P(x)}\,\forall\mathrm{I}^a?}{\vdash P(b)}\,\forall\mathrm{E}}{\vdash P(a) \supset P(b)}\,{\supset}\mathrm{I}^u}{\vdash \forall y.\, P(a) \supset P(y)}\,\forall\mathrm{I}^b}{\vdash \forall x.\, \forall y.\, P(x) \supset P(y)}\,\forall\mathrm{I}^a
$$

The flaw is at the inference marked with "?," where a is free in the hypothesis labelled u. Applying a local proof reduction to the (incorrect) $\forall$I inference followed by $\forall$E leads to the the assumption $[b/a]P(a)$ which is equal to $P(b)$. The resulting derivation

$$
\dfrac{\dfrac{\dfrac{\overline{\vdash P(b)}\,u}{\vdash P(a) \supset P(b)}\,{\supset}\mathrm{I}^u}{\vdash \forall y.\, P(a) \supset P(y)}\,\forall\mathrm{I}^b}{\vdash \forall x.\, \forall y.\, P(x) \supset P(y)}\,\forall\mathrm{I}^a
$$

is once again incorrect since the hypothesis labelled u should read $P(a)$, not $P(b)$.

The local expansion for universal quantification is much simpler.

$$
\begin{array}{c} \mathcal{D} \\ \vdash \forall x.\, A \end{array}
\quad \Longrightarrow_E \quad
\dfrac{\dfrac{\begin{array}{c} \mathcal{D} \\ \vdash \forall x.\, A \end{array}}{[a/x]A}\,\forall\mathrm{E}}{\vdash \forall x.\, A}\,\forall\mathrm{I}^a
$$

Existential Quantification. We conclude that $\exists x.\, A$ is true when there is a term t such that $[t/x]A$ is true.

$$
\dfrac{\vdash [t/x]A}{\vdash \exists x.\, A}\,\exists\mathrm{I}
$$

When we have an assumption $\exists x.\, A$ we do not know for which t it is the case that $[t/x]A$ holds. We can only assume that $[a/x]A$ holds for some parameter a about which we know nothing else. Thus the elimination rule resembles the one for disjunction.

$$\dfrac{\vdash \exists x.\ A \qquad \begin{array}{c}\overline{\vdash [a/x]A}^{\,u} \\ \vdots \\ \vdash C\end{array}}{\vdash C}\ \exists\mathrm{E}^{a,u}$$

The restriction is similar to the one for $\forall$I: the parameter a must be new, that is, it must not occur in $\exists x.\ A$, C, or any assumption employed in the derivation of the second premiss. In the reduction rule we have to perform two substitutions: we have to substitute t for the parameter a and we also have to substitute for the hypothesis labelled u.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D} \\ \vdash [t/x]A\end{array}}{\exists x.\ A}\ \exists\mathrm{I} \qquad \begin{array}{c}\overline{\vdash [a/x]A}^{\,u} \\ \mathcal{E} \\ \vdash C\end{array}}{\vdash C}\ \exists\mathrm{E}^{a,u} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D} \\ \overline{\vdash [t/x]A}^{\,u} \\ [t/a]\mathcal{E} \\ \vdash C\end{array}$$

The proviso on occurrences of a guarantees that the conclusion and hypotheses of $[t/a]\mathcal{E}$ have the correct form. The local expansion for existential quantification is also similar to the case for disjunction.

$$\begin{array}{c}\mathcal{D} \\ \vdash \exists x.\ A\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D} \\ \vdash \exists x.\ A\end{array} \qquad \dfrac{\overline{\vdash [a/x]A}^{\,u}}{\vdash \exists x.\ A}\ \exists\mathrm{I}}{\vdash \exists x.\ A}\ \exists\mathrm{E}^{a,u}$$

Here is a simple example of a natural deduction. We attempt to show the process by which such a deduction may have been generated, as well as the final deduction. The three vertical dots indicate a gap in the derivation we are trying to construct, with hypotheses and their consequences shown above and the desired conclusion below the gap.

$$\begin{array}{c}\vdots \\ \vdash A \wedge (A \supset B) \supset B\end{array} \quad \leadsto \quad \dfrac{\begin{array}{c}\overline{\vdash A \wedge (A \supset B)}^{\,u} \\ \vdots \\ \vdash B\end{array}}{\vdash A \wedge (A \supset B) \supset B}\ \supset\mathrm{I}^u$$

$$\rightsquigarrow \quad \dfrac{\begin{array}{c}\dfrac{\overline{\vdash A \wedge (A \supset B)}\;u}{\vdash A}\;\wedge\mathrm{E_L}\\ \vdots \\ \vdash B\end{array}}{\vdash A \wedge (A \supset B) \supset B}\;\supset\mathrm{I}^u \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\dfrac{\overline{\vdash A \wedge (A \supset B)}\;u}{\vdash A}\;\wedge\mathrm{E_L} \qquad \dfrac{\overline{\vdash A \wedge (A \supset B)}\;u}{\vdash A \supset B}\;\wedge\mathrm{E_R}\\ \vdots \\ \vdash B\end{array}}{\vdash A \wedge (A \supset B) \supset B}\;\supset\mathrm{I}^u$$

$$\rightsquigarrow \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\overline{\vdash A \wedge (A \supset B)}\;u}{\vdash A \supset B}\;\wedge\mathrm{E_R} \qquad \dfrac{\overline{\vdash A \wedge (A \supset B)}\;u}{\vdash A}\;\wedge\mathrm{E_L}}{\vdash B}\;\supset\mathrm{E}\\ \vdots \\ \vdash B\end{array}}{\vdash A \wedge (A \supset B) \supset B}\;\supset\mathrm{I}^u$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overline{\vdash A \wedge (A \supset B)}\;u}{\vdash A \supset B}\;\wedge\mathrm{E_R} \qquad \dfrac{\overline{\vdash A \wedge (A \supset B)}\;u}{\vdash A}\;\wedge\mathrm{E_L}}{\vdash B}\;\supset\mathrm{E}}{\vdash A \wedge (A \supset B) \supset B}\;\supset\mathrm{I}^u$$

The symbols A and B in this derivation stand for arbitrary propositions; we can thus established a judgment parametric in A and B. In other words, every instance of this derivation (substituting arbitrary propositions for A and B) is a valid derivation.

Below is a summary of the rules of intuitionistic natural deduction.

Introduction Rules

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\wedge\mathrm{I}$$

$$\frac{\vdash A}{\vdash A \vee B}\vee\mathrm{I_L} \qquad \frac{\vdash B}{\vdash A \vee B}\vee\mathrm{I_R}$$

$$\frac{\begin{array}{c}\overline{\vdash A}\,u\\ \vdots\\ \vdash B\end{array}}{\vdash A \supset B}\supset\mathrm{I}^{u}$$

$$\frac{\begin{array}{c}\overline{\vdash A}\,u\\ \vdots\\ \vdash p\end{array}}{\neg \vdash A}\neg\mathrm{I}^{p,u}$$

$$\frac{}{\vdash \top}\top\mathrm{I}$$

no $\bot$ introduction

$$\frac{\vdash [a/x]A}{\vdash \forall x.\ A}\forall\mathrm{I}^{a}$$

$$\frac{\vdash [t/x]A}{\vdash \exists x.\ A}\exists\mathrm{I}$$

Elimination Rules

$$\frac{\vdash A \wedge B}{\vdash A}\wedge\mathrm{E_L} \qquad \frac{\vdash A \wedge B}{\vdash B}\wedge\mathrm{E_R}$$

$$\frac{\vdash A \vee B \qquad \begin{array}{c}\overline{\vdash A}\,u_1\\ \vdots\\ \vdash C\end{array} \qquad \begin{array}{c}\overline{\vdash B}\,u_2\\ \vdots\\ \vdash C\end{array}}{\vdash C}\vee\mathrm{E}^{u_1,u_2}$$

$$\frac{\vdash A \supset B \qquad \vdash A}{\vdash B}\supset\mathrm{E}$$

$$\frac{\vdash A \qquad \vdash \neg A}{\vdash C}\neg\mathrm{E}$$

no $\top$ elimination

$$\frac{\vdash \bot}{\vdash C}\bot\mathrm{E}$$

$$\frac{\vdash \forall x.\ A}{\vdash [t/x]A}\forall\mathrm{E}$$

$$\frac{\vdash \exists x.\ A \qquad \begin{array}{c}\overline{\vdash [a/x]A}\,u\\ \vdots\\ \vdash C\end{array}}{\vdash C}\exists\mathrm{E}^{a,u}$$

1.2 Classical Logic

The inference rules so far only model *intuitionistic logic*, and some classically true propositions such as $A \vee \neg A$ (for an arbitrary A) are not derivable, as we will see in Section **??**. There are three commonly used ways one can construct a system of *classical natural deduction* by adding one additional rule of inference. $\perp_C$ is called *Proof by Contradiction* or *Rule of Indirect Proof*, $\neg\neg_C$ is the *Double Negation Rule*, and XM is referred to as *Excluded Middle*.

$$\dfrac{\begin{array}{c}\overline{\neg A}^{\,u}\\ \vdots \\ \perp\end{array}}{A}\perp_C^u \qquad \dfrac{\neg\neg A}{A}\neg\neg_C \qquad \dfrac{}{A \vee \neg A}\mathrm{XM}$$

The rule for classical logic (whichever one chooses to adopt) breaks the pattern of introduction and elimination rules. One can still formulate some reductions for classical inferences, but natural deduction is at heart an intuitionistic calculus. The symmetries of classical logic are much better exhibited in sequent formulations of the logic. In Exercise 1.3 we explore the three ways of extending the intuitionistic proof system and show that they are equivalent.

Another way to obtain a natural deduction system for classical logic is to allow multiple conclusions (see, for example, Parigot [Par92]).

1.3 Exercises

Exercise 1.1 Prove the following by natural deduction using only intuitionistic rules when possible. We use the convention that $\supset$, $\wedge$, and $\vee$ associate to the right, that is, $A \supset B \supset C$ stands for $A \supset (B \supset C)$. $A \equiv B$ is a syntactic abbreviation for $(A \supset B) \wedge (B \supset A)$. Also, we assume that $\wedge$ and $\vee$ bind more tightly than $\supset$, that is, $A \wedge B \supset C$ stands for $(A \wedge B) \supset C$. The scope of a quantifier extends as far to the right as consistent with the present parentheses. For example, $(\forall x.\, P(x) \supset C) \wedge \neg C$ would be disambiguated to $(\forall x.\, (P(x) \supset C)) \wedge (\neg C)$.

1. $\vdash A \supset B \supset A$.
2. $\vdash A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.
3. (Peirce's Law). $\vdash ((A \supset B) \supset A) \supset A$.
4. $\vdash A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.
5. $\vdash A \supset (A \wedge B) \vee (A \wedge \neg B)$.
6. $\vdash (A \supset \exists x.\, P(x)) \equiv \exists x.\, (A \supset P(x))$.
7. $\vdash ((\forall x.\, P(x)) \supset C) \equiv \exists x.\, (P(x) \supset C)$.

8. $\vdash \exists x.\, \forall y.\, (P(x) \supset P(y))$.

Exercise 1.2 We write $A \vdash B$ if B follows from hypothesis A and $A \dashv\vdash B$ for $A \vdash B$ and $B \vdash A$. Which of the following eight parametric judgments are derivable intuitionistically?

1. $(\exists x.\, A) \supset B \dashv\vdash \forall x.\, (A \supset B)$
2. $A \supset (\exists x.\, B) \dashv\vdash \exists x.\, (A \supset B)$
3. $(\forall x.\, A) \supset B \dashv\vdash \exists x.\, (A \supset B)$
4. $A \supset (\forall x.\, B) \dashv\vdash \forall x.\, (A \supset B)$

Provide natural deductions for the valid judgments. You may assume that the bound variable x does not occur in B (items 1 and 3) or A (items 2 and 4).

Exercise 1.3 Show that the three ways of extending the intuitionistic proof system are equivalent, that is, the same formulas are deducible in all three systems.

Exercise 1.4 Assume we had omitted disjunction and existential quantification and their introduction and elimination rules from the list of logical primitives. In the classical system, give a definition of disjunction and existential quantification (in terms of other logical constants) and show that the introduction and elimination rules now become *admissible rules of inference*. A rule of inference is *admissible* if any deduction using the rule can be transformed into one without using the rule.

Exercise 1.5 Assume we would like to design a system of natural deduction for a simple temporal logic. The main judgment is now "*A is true at time t*" written as

$$\vdash^t A.$$

1. Explain how to modify the given rules for natural deduction to this more general judgment and show the rules for implication and universal quantification.
2. Write out introduction and elimination rules for the temporal operator $\bigcirc A$ which should be true if A is true at the next point in time. Denote the "next time after t" by $t + 1$.
3. Show the local reductions and expansions which show the local soundness and completness of your rules.
4. Write out introduction and elimination rules for the temporal operator $\Box A$ which should be true if A is true at all times.
5. Show the local reductions and expansions.

Exercise 1.6 Design introduction and elimination rules for the connectives

1. $A \equiv B$, usually defined as $(A \supset B) \wedge (B \supset A)$,
2. $A \mid B$ (exclusive or), usually defined as $(A \wedge \neg B) \vee (\neg A \wedge B)$,

without recourse to other logical constants or operators. Also show the corresponding local reductions and expansions.

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