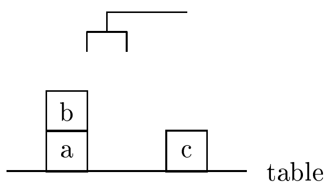
2.3 Two Examples

In this section we practice exploiting the connectives of linear logic to express situations involving resources and state. The first example is a *menu* consisting of various courses which can be obtained for 200 french francs.

Menu A: FF 200	$\mathrm{FF}(200) \multimap$
Onion Soup or *Clear Broth*	$((\mathrm{OS} \& \mathrm{CB})$
Honey-Glazed Duck	$\otimes\, \mathrm{HGD}$
Peas or *Red Cabbage* (according to season)	$\otimes\, (\mathrm{P} \oplus \mathrm{RC})$
New Potatoes	$\otimes\, \mathrm{NP}$
Chocolate Mousse (FF 30 extra)	$\otimes\, ((\mathrm{FF}(30) \multimap \mathrm{CM}) \& \mathbf{1})$
Coffee (unlimited refills)	$\otimes\, \mathrm{C}$ $\otimes\, (!\mathrm{C}))$

Note the two different informal uses of "or", one modelled by an alternative conjunction and one by a disjunction. The option of ordering chocolate mousse is also represented by an alternative conjunction: we can choose $(\mathrm{FF}(30) \multimap \mathrm{CM}) \& \mathbf{1}$ to obtain nothing ($\mathbf{1}$) or pay another 30 francs to obtain the mousse.

The second is perhaps more typical of uses of linear logic in computer science applications. We use it to model a planning problem in the so-called *blocks world* in which a robot arm can manipulate blocks, trying to achieve some goal.



We use the following primitive propositions.

$\mathrm{on}(x, y)$	block x is on block y
$\mathrm{tb}(x)$	block x is on the table
$\mathrm{holds}(x)$	robot arm holds block x
empty	robot arm is empty
$\mathrm{clear}(x)$	the top of block x is clear

A planning problem is represented as judgment

$$\Gamma_0; \Delta_0 \vdash A_0$$

where Γ_0 represent the rules which describe the legal operations, Δ_0 is the initial state represented as a context of the propositions which are true, and A is the goal to be achieved. For example, the situation in the picture above would be represented by

$$\Delta_0 = \cdot, \mathrm{empty}, \mathrm{tb}(a), \mathrm{on}(b,a), \mathrm{clear}(b), \mathrm{tb}(c), \mathrm{clear}(c)$$

where we have omitted labels for the sake of brevity. The rules are represented by unrestricted hypotheses, since they may be used arbitrarily often in the course of solving a problem. We use the following for rules for picking up or putting down and object. We use the convention that simultaneous conjunction $\otimes$ binds more tightly than linear implication $\multimap$.

$$\begin{array}{rcll}
\Gamma_0 & = & \cdot, & \\
 & & \mathrm{geton} & : \forall x.\, \forall y.\, \mathrm{empty} \otimes \mathrm{clear}(x) \otimes \mathrm{on}(x,y) \multimap \mathrm{holds}(x) \otimes \mathrm{clear}(y), \\
 & & \mathrm{gettb} & : \forall x.\, \mathrm{empty} \otimes \mathrm{clear}(x) \otimes \mathrm{tb}(x) \multimap \mathrm{holds}(x), \\
 & & \mathrm{puton} & : \forall x.\, \forall y.\, \mathrm{holds}(x) \otimes \mathrm{clear}(y) \multimap \mathrm{empty} \otimes \mathrm{on}(x,y) \otimes \mathrm{clear}(x), \\
 & & \mathrm{puttb} & : \forall x.\, \mathrm{holds}(x) \multimap \mathrm{empty} \otimes \mathrm{tb}(x) \otimes \mathrm{clear}(x).
\end{array}$$

Each of these represents a particular possible action, assuming that it can be carried out successfully. Matching the left-hand side of one these rules will consume the corresponding resources so that, for example, the proposition *empty* with no longer be available after the *geton* action has been applied.

The goal that we would like to achieve $\mathrm{on}(a,b)$, for example, is represented with the aid of using $\top$.

$$A_0 = \mathrm{on}(a,b) \otimes \top$$

Any derivation of the judgment

$$\Gamma_0; \Delta_0 \vdash A_0$$

represents a plan for achieving the goal A_0 from the initial situation state Δ_0.

We now go through a derivation of the particular example above, omitting the unrestricted resources Γ_0 which do not change throughout the derivation. Our first goal is to derive

$$\cdot, \mathrm{empty}, \mathrm{tb}(a), \mathrm{on}(b,a), \mathrm{clear}(b), \mathrm{tb}(c), \mathrm{clear}(c), \mathrm{empty} \vdash \mathrm{on}(a,b) \otimes \top$$

By using $\otimes$I twice we can prove

$$\cdot, \mathrm{empty}, \mathrm{on}(b,a), \mathrm{clear}(b) \vdash \mathrm{empty} \otimes \mathrm{clear}(b) \otimes \mathrm{on}(b,a)$$

Using the intuitionistic hypothesis rule for *geton* followed by $\forall$E twice and $\multimap$E we obtain

$$\cdot, \mathrm{empty}, \mathrm{clear}(b), \mathrm{on}(b,a) \vdash \mathrm{holds}(b) \otimes \mathrm{clear}(a)$$

Now we use $\otimes$E with the derivation above as our left premiss, to prove our overall goal, leaving us with the goal to derive

$$\cdot, \mathrm{tb}(a), \mathrm{tb}(c), \mathrm{clear}(c), \mathrm{holds}(b), \mathrm{clear}(a) \vdash \mathrm{on}(a, b) \otimes \top$$

as our right premiss. Observe how the original resources Δ_0 have been split between the two premisses, and the results from the left premiss derivation, $\mathrm{holds}(b)$ and $\mathrm{clear}(a)$ have been added to the description of the situation. The new subgoal has exactly the same form as the original goal (in fact, the conclusion has not changed), but applying the unrestricted assumption *geton* has changed our state.

Proceeding in the same manner, using the rule *puttb* next leaves us with the subgoal

$$\cdot, \mathrm{tb}(a), \mathrm{tb}(c), \mathrm{clear}(c), \mathrm{clear}(a), \mathrm{empty}, \mathrm{clear}(b), \mathrm{tb}(b) \vdash \mathrm{on}(a, b) \otimes \top$$

We now apply *gettb* using a for x and proceeding as above which gives us a derivation of $\vdash \mathrm{holds}(a)$. Instead of $\otimes$E, we use the substitution principle yielding the subgoal

$$\cdot, \mathrm{tb}(c), \mathrm{clear}(c), \mathrm{clear}(b), \mathrm{tb}(b), \mathrm{holds}(a) \vdash \mathrm{on}(a, b) \otimes \top$$

With same technique, this time using *puton*, we obtain the subgoal

$$\cdot, \mathrm{tb}(c), \mathrm{clear}(c), \mathrm{tb}(b), \mathrm{empty}, \mathrm{on}(a, b), \mathrm{clear}(a) \vdash \mathrm{on}(a, b) \otimes \top$$

Now we can conclude the derivation with the $\otimes$I rule, distributing resource $\mathrm{on}(a, b)$ to the left premiss, which follows immediately as hypothesis, and distributing the remaining resources to the right premiss, where $\top$ follows by $\top$I, ignoring all resources.

Note that different derivations of the original judgment represent different sequences of actions (see Exercise 2.4).

2.4 Embedding Intuitionistic Logic

Our goal in this section is to show that intuitionistic linear logic (ILL) is a refinement of intuitionistic logic (IL) in the sense that we can translate each formula of IL into ILL in a way that preserves derivability. Actually, we will try to achieve more: not only should it be possible to preserve derivability, but the translation should also preserve the structure of derivations as much as possible. This will allow us to make stronger statements regarding the connection between proof search and reduction in the two calculi when we investigate specific applications.

The guiding principle in the definition of the translation $()^+$ of IL formulas into ILL is the idea that the judgment $\Gamma \vdash A$ of IL is interpreted as the judgment $\Gamma^+; \cdot \vdash A^+$ of ILL. In other words, all intuitionistic assumptions become unrestricted hypotheses. We design the translation so that a derivation $\mathcal{D} :: (\Gamma \vdash A)$

can be translated directly to a derivation $\mathcal{D}^+ :: (\Gamma^+; \cdot \vdash A^+)$. We omit negation here, which is left to Exercise 2.10.

The only real question arises in the cases for conjunction and truth, since they split into two possible connectives each. We model them here with additive linear connectives. Another translation is explored in Exercise 2.9. For most connectives, however, we have little choice. Contexts are translated by simply translating the formulas occurring in them.

$$
\begin{array}{rcl}
P^+ & = & P \\
(A \wedge B)^+ & = & A^+ \& B^+ \\
(A \supset B)^+ & = & (!A^+) \multimap B^+ \\
(A \vee B)^+ & = & (!A^+) \oplus (!B^+) \\
(\bot)^+ & = & \mathbf{0} \\
(\top)^+ & = & \top \\
(\forall x.\, A)^+ & = & \forall x.\, A^+ \\
(\exists x.\, A)^+ & = & \exists x.\, !A^+ \\
(\cdot)^+ & = & \cdot \\
(\Gamma, u{:}A)^+ & = & \Gamma^+, u{:}A^+
\end{array}
$$

To illustrate Girard's original decomposition of $A \supset B$ into $(!A) \multimap B$ we do not use intuitionistic implication in linear logic, even though it would certainly be reasonable to translate $(A \supset B)^+ = A^+ \supset B^+$.

Lemma 2.1 (Embedding) *If $\Gamma \vdash A$ in IL then $\Gamma^+; \cdot \vdash A^+$ in ILL.*

Proof: By induction over the structore of $\mathcal{D} :: (\Gamma \vdash A)$. The computational contents of this proof is a compositional translation of derivations $\mathcal{D}$ to derivations $\mathcal{D}^+ :: (\Gamma^+; \cdot \vdash A^+)$. □

An attempt to prove the other direction in a similar manner will fail, since a natural deduction of $\Gamma^+; \cdot \vdash A^+$ may have many subdeductions with a conclusion which is not of this form. For example, if the deduction ends in $\multimap E$ the premises contain the new formula A which may not necessarily be the translation of an intuitionistic formula. In the next section we show a way to prove the opposite direction based on normal derivations. A simpler way is to translate each linear connective into its intuitionistic counterpart and show that the resulting judgment is derivable. This reverse translation $()^-$ should have the property that $(A^+)^- = A$

$$
\begin{array}{rcl}
P^- & = & P \\
(A \& B)^- = (A \otimes B)^- & = & A^- \wedge B^- \\
(A \multimap B)^- = (A \supset B)^- & = & A^- \supset B^- \\
(A \oplus B)^- & = & A^- \vee B^- \\
(\mathbf{0})^- & = & \perp \\
(\top)^- = (\mathbf{1})^- & = & \top \\
(\forall x.\, A)^- & = & \forall x.\, A^- \\
(\exists x.\, A)^- & = & \exists x.\, A^- \\
(!A)^- & = & A^- \\
\\
(\cdot)^- & = & \cdot \\
(\Gamma, u{:}A)^- & = & \Gamma^-, u{:}A^-
\end{array}
$$

The last two rules are also used to map a linear context Δ to the corresponding intuitionistic context Δ^-.

Property 2.2 $(A^+)^- = A$

Proof: By induction on the structure of A. □

Lemma 2.3 (Conservativity) *If* $\Gamma; \Delta \vdash A$ *in ILL then* $\Gamma^-, \Delta^- \vdash A^-$ *in IL.*

Proof: By induction on the structure of $\mathcal{D} :: (\Gamma; \Delta \vdash A)$. □

Theorem 2.4 (Conservative Embedding) *The translation* $()^+$ *is a conservative embedding from IL into ILL.*

Proof: From Lemmas 2.1 and 2.3 and Property 2.2. □

2.5 Exercises

Exercise 2.1 Give a counterexample which shows that the elimination $\supset$E would be locally unsound if its second premiss were allowed to depend on linear hypotheses.

Exercise 2.2 If we *define* intuitionistic implication $A \supset B$ in linear logic as an abbreviation for $(!A) \multimap B$, then the given introduction and elimination rules become *derived rules of inference.* Prove this by giving a derivation for the conclusion of the $\supset$E rule from its premisses under the interpretation, and similarly for the $\supset$I rule.

For the other direction, show how $!A$ could be defined from intuitionistic implication or speculate why this might not be possible.

Exercise 2.3 [*To be filled in: an exercise exploring the "missing connectives" of multiplicative disjunction and additive implication.*]

Exercise 2.4 In the blocks world example from Section 2.3, sketch the derivation for the same goal A_0 and initial situation Δ_0 in which block b is put on block c, rather than the table.

Exercise 2.5 Model the *Towers of Hanoi* in linear logic in analogy with our modelling of the blocks world.

1. Define the necessary atomic propositions and their meaning.
2. Describe the legal moves in *Towers of Hanoi* as unrestricted hypotheses Γ_0 independently from the number of towers or disks.
3. Represent the initial situation of three towers, where two are empty and one contains two disks in a legal configuration.
4. Represent the goal of legally stacking the two disks on some arbitrary other tower.
5. Sketch the proof for the obvious 3-move solution as in Section 2.3.

Exercise 2.6 Consider if $\otimes$ and $\&$ can be distributed over $\oplus$ or *vice versa*. There are four different possible equivalences based on eight possible entailments. Give natural deductions for the entailments which hold.

Exercise 2.7 In this exercise we explore distributive and related *interaction laws* for linear implication. In intuitionistic logic, for example, we have the following $(A \wedge B) \supset C \dashv\vdash A \supset (B \supset C)$ and $A \supset (B \wedge C) \dashv\vdash (A \supset B) \wedge (A \supset C)$, where $\dashv\vdash$ is mutual entailment as in Exercise 1.2.

In linear logic, we now write $A \dashv\vdash A'$ for linear mutual entailment, that is, A' follows from linear hypothesis A and *vice versa*. Write out appropriate interaction laws or indicate none exists, for each of the following propositions.

1. $A \multimap (B \otimes C)$
2. $(A \otimes B) \multimap C)$
3. $A \multimap \mathbf{1}$
4. $\mathbf{1} \multimap A$
5. $A \multimap (B \& C)$
6. $(A \& B) \multimap C$
7. $A \multimap \top$
8. $\top \multimap A$
9. $A \multimap (B \oplus C)$
10. $(A \oplus B) \multimap C$

11. $A \multimap \mathbf{0}$

12. $\mathbf{0} \multimap A$

13. $A \multimap (B \multimap C)$

14. $(A \multimap B) \multimap C$

Note that an interaction law exists only if there is a mutual linear entailment—we are not interested if one direction holds, but not the other.

Give the derivations in both directions for one of the interaction laws of a binary connective $\otimes$, $\&$, $\oplus$, or $\multimap$, and for one of the interaction laws of a logical constant $\mathbf{1}$, $\top$, or $\mathbf{0}$.

Exercise 2.8 Extend the interaction laws from Exercise 2.7 by laws showing how linear implication interacts with existential and universal quantification.

Exercise 2.9 Design an alternative translation $()^*$ from formulas and natural deductions in intuitionistic logic to intuitionistic linear logic in which conjunction ($\wedge$) and truth ($\top$) are mapped to simultaneous conjunction ($\otimes$) and its unit ($\mathbf{1}$) instead of the additive connectives as in $()^+$. Prove the correctness of the embedding and discuss the relative merits of the two translations.

Exercise 2.10 Extend the embedding from from Section 2.4 to encompass intuitionistic propositions $\neg A$ without adding any connectives to the linear logic. Modify the statements and proofs of embedding and conservativity (if necessary) and show the proof cases concerned with negation.