Linear Logic 15-816: Homework 1

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1.2 Natural Deductions

In the derivations below we do not label the undischarged hypotheses so they are easier to discern in the derivation. We tacitly use the assumptions about variable occurrences to calculate substitutions.

1a. $(\exists x.A) \supset B \vdash \forall x.(A \supset B)$**,** x **not free in** B

$$\dfrac{\dfrac{\dfrac{\vdash (\exists x.A) \supset B \qquad \dfrac{\dfrac{}{\vdash [a/x](A)}\,u}{\vdash \exists x.A}\,\exists\mathrm{I}}{\vdash B}\supset \mathrm{E}}{\vdash [a/x](A \supset B)}\supset \mathrm{I}^u}{\vdash \forall x.(A \supset B)}\,\forall\mathrm{I}^a$$

1b. $(\exists x.A) \supset B \dashv \forall x.(A \supset B)$**,** x **not free in** B

$$\dfrac{\dfrac{\dfrac{}{\vdash \exists x.A}\,u \qquad \dfrac{\dfrac{\vdash \forall x.(A \supset B)}{\vdash ([a/x]A) \supset B}\,\forall\mathrm{E} \qquad \dfrac{}{\vdash [a/x]A}\,w}{\vdash B}\supset \mathrm{E}}{\vdash B}\,\exists\mathrm{E}^{a,w}}{\vdash (\exists x.A) \supset B}\supset \mathrm{I}^u$$

2a. $A \supset (\exists x.B) \vdash \exists x.(A \supset B)$**,** x **not free in** A

Not derivable in intuitionistic logic.

2b. $A \supset (\exists x.B) \dashv \exists x.(A \supset B)$**,** x **not free in** A

$$\dfrac{\vdash \exists x.(A \supset B) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A}\,u \qquad \dfrac{}{\vdash A \supset [a/x]B}\,v}{\vdash [a/x]B}\supset \mathrm{E}}{\vdash \exists x.B}\,\exists\mathrm{I}}{\vdash A \supset (\exists x.B)}\supset \mathrm{I}^u}{\vdash A \supset (\exists x.B)}\,\exists\mathrm{E}^{a,v}$$

3a. $(\forall x.A) \supset B \vdash \exists x.(A \supset B)$**,** x **not free in** B

Not derivable in intuitionistic logic.

3b. $(\forall x.A) \supset B \dashv \exists x.(A \supset B)$**,** x **not free in** B

$$\dfrac{\dfrac{\vdash \exists x.(A \supset B) \qquad \dfrac{\dfrac{}{\vdash [a/x](A) \supset B}\,v \qquad \dfrac{\dfrac{}{\vdash (\forall x.A)}\,u}{\vdash [a/x]A}\,\forall\mathrm{E}}{\vdash B}\supset \mathrm{E}}{\vdash B}\,\exists\mathrm{E}^{a,v}}{\vdash (\forall x.A) \supset B}\supset \mathrm{I}^u$$

4a. $A \supset (\forall x.B) \vdash \forall x.(A \supset B)$**, x not free in A**

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash A \supset (\forall x.B) \qquad \cfrac{}{\vdash A}\,u}{\vdash \forall x.B}\supset\mathrm{E}}{\vdash [a/x]B}\forall\mathrm{E}}{\vdash A \supset [a/x]B}\supset\mathrm{I}^u}{\vdash \forall x.(A \supset B)}\forall\mathrm{I}^a$$

4b. $A \supset (\forall x.B) \dashv \forall x.(A \supset B)$**, x not free in A**

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash \forall x.(A \supset B)}{\vdash A \supset [a/x]B}\forall\mathrm{E} \qquad \cfrac{}{\vdash A}\,u}{\vdash [a/x]B}\supset\mathrm{E}}{\vdash \forall x.B}\forall\mathrm{I}^a}{\vdash A \supset (\forall x.B)}\supset\mathrm{I}^u$$

1.5 Temporal Logic

1. To modify rules for a simple temporal logic, change $\vdash$ to $\vdash^t$ in all cases. Examples follow:

$$\cfrac{\begin{matrix}\cfrac{}{\vdash^t A}\,u \\ \vdots \\ \vdash^t B\end{matrix}}{\vdash^t A \supset B}\supset\mathrm{I}^u \qquad \cfrac{\vdash^t A \supset B \qquad \vdash^t A}{\vdash^t B}\supset\mathrm{E} \qquad \cfrac{\vdash^t [a/x]A}{\vdash^t \forall x.A}\forall\mathrm{I} \quad \cfrac{\vdash^t \forall x.A}{\vdash^t [s/x]A}\forall\mathrm{E}$$

2. Introduction and elimination rules for temporal operator $\bigcirc A$:

$$\cfrac{\vdash^{t+1} A}{\vdash^t \bigcirc A}\bigcirc\mathrm{I} \qquad\qquad \cfrac{\vdash^t \bigcirc A}{\vdash^{t+1} A}\bigcirc\mathrm{E}$$

3. Local reduction and expansion for $\bigcirc A$:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \vdash^{t+1} A\end{matrix}}{\vdash^t \bigcirc A}\bigcirc\mathrm{I}}{\vdash^{t+1} A}\bigcirc\mathrm{E} \quad \Rightarrow_R \quad \begin{matrix}\mathcal{D}\\ \vdash^{t+1} A\end{matrix} \qquad\qquad \begin{matrix}\mathcal{D}\\ \vdash^t \bigcirc A\end{matrix} \quad \Rightarrow_E \quad \cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \vdash^t \bigcirc A\end{matrix}}{\vdash^{t+1} A}\bigcirc\mathrm{E}}{\vdash^t \bigcirc A}\bigcirc\mathrm{I}$$

4. Introduction and elimination rules for temporal operator $\Box A$:

$$\cfrac{\vdash^a A}{\vdash^t \Box A}\Box\mathrm{I}^a \qquad\qquad \cfrac{\vdash^t \Box A}{\vdash^s A}\Box\mathrm{E}$$

   In the introduction rule a is a new parameter ranging over time. In the elimination rule, s is an arbitrary time. Note that these rules are completely generic and do not assume anything about the structure of time. For example, if we assumed that time has the structure of the natural numbers or the integers, we could use rules which allow us to conclude $\Box A$ in more circumstances.

5. Local reduction and expansion for $\Box A$:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \vdash^a A\end{matrix}}{\vdash^t \Box A}\Box\mathrm{I}^a}{\vdash^s A}\Box\mathrm{E} \quad \Rightarrow_R \quad \begin{matrix}[s/a]\mathcal{D}\\ \vdash^s A\end{matrix} \qquad\qquad \begin{matrix}\mathcal{D}\\ \vdash^t \Box A\end{matrix} \quad \Rightarrow_E \quad \cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \vdash^t \Box A\end{matrix}}{\vdash^a A}\Box\mathrm{E}}{\vdash^t \Box A}\Box\mathrm{I}^a$$