RECITATION 11: FULL FOCUSING, Q&A

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We will spend the first half of recitation revisiting focusing. The second part of the recitation will be dedicated to answering questions about tomorrow's midterm[1].

1. FOCUSING

We begin by reminding ourselves about polarities. A *negative proposition* is one with an invertible right rule. A *positive proposition* is one with an invertible left rule. See the handout for a refresher on all of the rules.

We begin by exploring how changing the polarity of atoms can drastically change the shape of a proof. Consider the proposition $a \vee b \wedge c \supset (a \vee b) \wedge (a \vee c)$. Suppose we first made every atomic proposition positive:[2]

$$a^+ \vee (b^+ \wedge c^+) \supset \uparrow((a^+ \vee b^+) \wedge (a^+ \vee c^+)).$$

For compactness on the board (and on this page), we will write X^+ for $(a^+ \vee b^+) \wedge (a^+ \vee c^+)$. The proof would then look like this:

$$\dfrac{\dfrac{\dfrac{}{a^+ \longrightarrow [a+]}}{a^+ \longrightarrow [a^+ \vee b^+]} \quad \dfrac{\dfrac{}{a^+ \longrightarrow [a+]}}{a^+ \longrightarrow [a^+ \vee c^+]}}{\dfrac{a^+ \longrightarrow [X^+]}{\dfrac{a^+ \longrightarrow X^+}{\dfrac{a^+ ; \bullet \xrightarrow{L} X^+}{\bullet ; a^+ \xrightarrow{L} X^+}}}} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{b^+, c^+ \longrightarrow [b^+]}}{b^+, c^+ \longrightarrow [a^+ \vee b^+]} \quad \dfrac{\dfrac{}{b^+, c^+ \longrightarrow [c^+]}}{b^+, c^+ \longrightarrow [a^+ \vee c^+]}}{b^+, c^+ \longrightarrow [X^+]}}{\dfrac{b^+, c^+ \longrightarrow X^+}{\dfrac{b^+, c^+ ; \bullet \xrightarrow{L} X^+}{\dfrac{c^+ ; b^+ \xrightarrow{L} X^+}{\dfrac{\bullet ; b^+ \cdot c^+ \xrightarrow{L} X^+}{\bullet ; b^+ \wedge c^+ \xrightarrow{L} X^+}}}}}$$

$$\dfrac{\dfrac{\bullet ; a^+ \vee (b^+ \wedge c^+) \xrightarrow{L} X^+}{\bullet ; a^+ \vee (b^+ \wedge c^+) \xrightarrow{R} \uparrow X^+}}{\bullet ; \bullet \xrightarrow{R} a^+ \vee (b^+ \wedge c^+) \supset \uparrow X^+}$$

Now suppose we made every atomic proposition negative:

$$\downarrow a^- \vee \downarrow(b^- \wedge c^-) \supset \uparrow(\downarrow a^- \vee \downarrow b^-) \wedge \uparrow(\downarrow a^- \vee \downarrow c^-).$$

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[1] Remind students that they have a midterm tomorrow.

[2] Consider having students place the arrows in these to make sure they understand what's going on.

Then we would get the following very different proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[a^-] \longrightarrow a^-}}{a^- \longrightarrow a^-}}{a^- ; \bullet \xrightarrow{R} a^-}}{a^- \longrightarrow [\downarrow a^-]}}{a^- \longrightarrow [\downarrow a^- \vee \downarrow b^-]}}{a^- \longrightarrow \downarrow a^- \vee \downarrow b^-}}{a^- ; \bullet \xrightarrow{L} \downarrow a^- \vee \downarrow b^-} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[b^-] \longrightarrow b^-}}{[b^- \wedge c^-] \longrightarrow b^-}}{b^- \wedge c^- \longrightarrow b^-}}{b^- \wedge c^- ; \bullet \xrightarrow{R} b^-}}{b^- \wedge c^- \longrightarrow [\downarrow b^-]}}{b^- \wedge c^- \longrightarrow [\downarrow a^- \vee \downarrow b^-]}}{b^- \wedge c^- \longrightarrow \downarrow a^- \vee \downarrow b^-}}{b^- \wedge c^- ; \bullet \xrightarrow{L} \downarrow a^- \vee \downarrow b^-}}{\bullet ; \downarrow a^- \xrightarrow{L} \downarrow a^- \vee \downarrow b^- \qquad \bullet ; \downarrow (b^- \wedge c^-) \xrightarrow{L} \downarrow a^- \vee \downarrow b^-}$$

$$\dfrac{\bullet ; \downarrow a^- \vee \downarrow (b^- \wedge c^-) \xrightarrow{R} \uparrow (\downarrow a^- \vee \downarrow b^-) \qquad \textit{symmetric case}}{\bullet ; \downarrow a^- \vee \downarrow (b^- \wedge c^-) \xrightarrow{R} \uparrow (\downarrow a^- \vee \downarrow b^-) \wedge \uparrow (\downarrow a^- \vee \downarrow c^-)}$$

$$\bullet ; \bullet \xrightarrow{R} \downarrow a^- \vee \downarrow (b^- \wedge c^-) \supset \uparrow (\downarrow a^- \vee \downarrow b^-) \wedge \uparrow (\downarrow a^- \vee \downarrow c^-)$$

If students would like to do an examples with atomic propositions of different polarities, do so.[3]

We now turn to an example involving forward chaining[4] and then find the corresponding derived rules of inference. Consider the proposition

$$a \supset ((a \vee b) \supset c) \supset (c \supset d) \supset d.$$

After applying right inversion sufficiently many times, one reaches

$$a, (a \vee b) \supset c, c \supset d \xrightarrow{R} d.$$

We make all of the atoms positive to set ourselves up for forward chaining:

$$a^+, (a^+ \vee b^+) \supset \uparrow c^+, c^+ \supset \uparrow d^+ \xrightarrow{R} d^+.$$

[3] This will, however, cut into the Q&A time for reviewing midterm material.

[4] We work in the forward chaining fragment of Lecture 18; this may be confusing after seeing all of the focusing rules. If the students prefer, amend the examples below to use those rules instead.

The proof then looks like follows, where $\Gamma_0 = \{a^+, (a^+ \vee b^+) \supset \uparrow c^+, c^+ \supset \uparrow d^+\}$.

$$\dfrac{\dfrac{a^+ \in \Gamma_0}{\Gamma_0 \xrightarrow{f} [a^+]}}{\Gamma_0 \xrightarrow{f} [a^+ \vee b^+]} \qquad \dfrac{\dfrac{}{\Gamma_0, c^+ \xrightarrow{f} [c^+]} \qquad \dfrac{\dfrac{}{\Gamma_0, c^+, d^+ \xrightarrow{f} [d^+]}}{\Gamma_0, c^+, d^+ \xrightarrow{f} d^+}}{\Gamma_0, c^+, [\uparrow d^+] \xrightarrow{f} d^+}}{\dfrac{\Gamma_0, c^+, [c^+ \supset \uparrow d^+] \xrightarrow{f} d^+}{\Gamma_0, c^+ \xrightarrow{f} d^+}}$$

$$\dfrac{\Gamma_0, [\uparrow c^+] \xrightarrow{f} d^+}{\Gamma_0, [(a^+ \vee b^+) \supset \uparrow c^+] \xrightarrow{f} d^+}$$

$$\Gamma_0 \xrightarrow{f} d^+$$

To find the derived rules of inference, we remark that we can only focus on two propositions in Γ_0, namely, $(a^+ \vee b^+) \supset \uparrow c^+$ and $c^+ \supset \uparrow d^+$. Let's focus on these in turn. Remark that we always go from stable sequents to stable sequents in deriving rules of inference.

$$\dfrac{\dfrac{\dfrac{a^+ \in \Gamma}{\Gamma \xrightarrow{f} [a^+]}}{\Gamma \xrightarrow{f} [a^+ \vee b^+]} \qquad \dfrac{\Gamma, c^+ \xrightarrow{f} \rho}{\Gamma, [\uparrow c^+] \xrightarrow{f} \rho}}{\Gamma, [(a^+ \vee b^+) \supset \uparrow c^+] \xrightarrow{f} \rho}$$

We then get the derived rule

$$\dfrac{\Gamma, a^+, c^+ \xrightarrow{f} \rho}{\Gamma, a^+ \xrightarrow{f} \rho}\ R_0$$

By symmetry, we also get

$$\dfrac{\Gamma, b^+, c^+ \xrightarrow{f} \rho}{\Gamma, b^+ \xrightarrow{f} \rho}\ R_0'$$

Focusing on $c^+ \supset \uparrow d^+$ gives us:

$$\dfrac{\dfrac{c^+ \in \Gamma}{\Gamma \xrightarrow{f} [c^+]} \qquad \dfrac{\Gamma, d^+ \xrightarrow{f} \rho}{\Gamma, [\uparrow d^+] \xrightarrow{f} \rho}}{\Gamma, [c^+ \supset \uparrow d^+] \xrightarrow{f} \rho}$$

which gives the rule

$$\dfrac{\Gamma, c^+, d^+ \xrightarrow{f} \rho}{\Gamma, c^+ \xrightarrow{f} \rho}\ R_1$$

Finally, we can focus on the succedent:

$$\dfrac{\dfrac{d^+ \in \Gamma}{\Gamma \xrightarrow{f} [d^+]}}{\Gamma \xrightarrow{f} d^+}$$

to get the derived rule

$$\dfrac{}{\Gamma, d^+ \xrightarrow{f} d^+}\ R_2$$

The original proof can then be simplified to

$$\dfrac{\dfrac{\dfrac{}{\Gamma_0, c^+, d^+ \xrightarrow{f} d^+}\ R_2}{\Gamma_0, c^+ \xrightarrow{f} d^+}\ R_1}{\Gamma_0 \xrightarrow{f} d^+}\ R_0$$

2. Midterm Q&A

Answer questions students may have about the material.