Constructive Logic (15-317), Fall 2017
Recitation 10: Focused Proof Search

November 1, 2017

1 Focused Proof Search

In this recitation we will focus[1] on an alternative proof search strategy for the negative fragment of intuitionistic propositional logic. Previously we have sought to reduce the uncertainty by making the logic more and more restricted without decreasing its power. This has taken us from natural deduction, to the sequent calculus, to a version of the sequent calculus where we apply inversion rules first and finally, now, to focusing. The idea is similar to that behind the inversion strategy, occasionally in the logic we must guess but there are many rules which are always safe to apply. As before we will greedily apply those rules but, unlike before, we will also consolidate our guesses. That is, whenever we're forced to make a guess we make as many guesses as possible. By batching guesses like this, we avoid even more nondeterminism.

The inference rules for this logic are as follows. First the rules that handle inversion. Since we're dealing with negative propositions, they all are all right rules and constitute all the right rules in the system.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R \qquad \frac{}{\Gamma \longrightarrow \top} \top R$$

Next comes the rule for selecting a proposition to focus on.

$$\frac{A \in \Gamma \quad \Gamma; [A] \longrightarrow P}{\Gamma \longrightarrow P} \,\mathsf{focus}$$

Then we have the batched rules for guesses.

$$\frac{\Gamma \longrightarrow A \quad \Gamma; [B] \longrightarrow P}{\Gamma; [A \supset B] \longrightarrow P} \supset L \qquad \frac{\Gamma; [A] \longrightarrow P}{\Gamma; [A \wedge B] \longrightarrow P} \wedge L1 \qquad \frac{\Gamma; [B] \longrightarrow P}{\Gamma; [A \wedge B] \longrightarrow P} \wedge L2 \qquad \frac{}{\Gamma; [P] \longrightarrow P} \wedge P$$

Notice in particular, the lack of rules for $P \neq Q$ when focusing on Q. This means that if focusing on a goal does not produce the result, we backtrack and focus on a new proposition. Now for some examples.

[1] Heh.

1. $((P \land Q) \land R) \implies (P \land (Q \land R))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \land Q) \land R; [P] \longrightarrow P}}{(P \land Q) \land R; [P \land Q] \longrightarrow P}}{(P \land Q) \land R; [(P \land Q) \land R] \longrightarrow P}}{(P \land Q) \land R \longrightarrow P} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \land Q) \land R; [Q] \longrightarrow Q}}{(P \land Q) \land R; [P \land Q] \longrightarrow Q}}{(P \land Q) \land R; [(P \land Q) \land R] \longrightarrow Q}}{(P \land Q) \land R \longrightarrow Q} \quad \dfrac{\dfrac{\dfrac{}{(P \land Q) \land R; [R] \longrightarrow R}}{(P \land Q) \land R; [(P \land Q) \land R] \longrightarrow P}}{(P \land Q) \land R \longrightarrow R}}{(P \land Q) \land R \longrightarrow Q \land R}}{(P \land Q) \land R \longrightarrow Q \land R}}{(P \land Q) \land R \longrightarrow P \land (Q \land R)}}{\cdot \longrightarrow ((P \land Q) \land R) \implies (P \land (Q \land R))}$$

2. $(P \supset Q \supset R) \supset (P \supset Q) \supset P \supset R$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \supset Q \supset R), (P \supset Q), P; [P] \longrightarrow P}}{(P \supset Q \supset R), (P \supset Q), P \longrightarrow P} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \supset Q \supset R), (P \supset Q), P; [P] \longrightarrow P}}{(P \supset Q \supset R), (P \supset Q), P \longrightarrow P} \quad \dfrac{}{(P \supset Q \supset R), (P \supset Q), P; [Q] \longrightarrow Q}}{(P \supset Q \supset R), (P \supset Q), P; [P \supset Q] \longrightarrow Q}}{(P \supset Q \supset R), (P \supset Q), P \longrightarrow Q} \quad \dfrac{}{(P \supset Q \supset R), (P \supset Q), P; [R] \longrightarrow R}}{(P \supset Q \supset R), (P \supset Q), P; [Q \supset R] \longrightarrow R}}{(P \supset Q \supset R), (P \supset Q), P; [P \supset Q \supset R] \longrightarrow R}}{(P \supset Q \supset R), (P \supset Q), P \longrightarrow R}}{(P \supset Q \supset R), (P \supset Q) \longrightarrow P \supset R}}{(P \supset Q \supset R) \longrightarrow (P \supset Q) \supset P \supset R}}{\cdot \longrightarrow (P \supset Q \supset R) \supset (P \supset Q) \supset P \supset R}$$

3. $(P \supset Q \supset R) \supset (P \land Q) \supset R$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \supset Q \supset R), (P \land Q); [P] \longrightarrow P}}{(P \supset Q \supset R), (P \land Q); [P \land Q] \longrightarrow P}}{(P \supset Q \supset R), (P \land Q) \longrightarrow P} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{(P \supset Q \supset R), (P \land Q); [Q] \longrightarrow Q}}{(P \supset Q \supset R), (P \land Q); [P \land Q] \longrightarrow Q}}{(P \supset Q \supset R), (P \land Q) \longrightarrow Q} \quad \dfrac{}{(P \supset Q \supset R), (P \land Q); [R] \longrightarrow R}}{(P \supset Q \supset R), (P \land Q); [Q \supset R] \longrightarrow R}}{(P \supset Q \supset R), (P \land Q); [P \supset Q \supset R] \longrightarrow R}}{(P \supset Q \supset R), (P \land Q) \longrightarrow R}}{(P \supset Q \supset R), (P \land Q) \longrightarrow R}}{(P \supset Q \supset R) \longrightarrow (P \land Q) \supset R}}{\cdot \longrightarrow (P \supset Q \supset R) \supset (P \land Q) \supset R}$$

4. $((P \land Q) \supset R) \supset P \supset (Q \supset R)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((P \land Q) \supset R), P, Q; [P] \longrightarrow P}}{((P \land Q) \supset R), P, Q \longrightarrow P} \quad \dfrac{\dfrac{}{((P \land Q) \supset R), P, Q; [Q] \longrightarrow Q}}{((P \land Q) \supset R), P, Q \longrightarrow Q}}{((P \land Q) \supset R), P, Q \longrightarrow P \land Q} \quad \dfrac{}{((P \land Q) \supset R), P, Q; [R] \longrightarrow R}}{((P \land Q) \supset R), P, Q; [(P \land Q) \supset R] \longrightarrow R}}{((P \land Q) \supset R), P, Q \longrightarrow R}}{((P \land Q) \supset R), P \longrightarrow (Q \supset R)}}{((P \land Q) \supset R) \longrightarrow P \supset (Q \supset R)}}{\cdot \longrightarrow ((P \land Q) \supset R) \supset P \supset (Q \supset R)}$$