

# Constructive Logic (15-317), Fall 2017

## Assignment 3: Verification and Quantification

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Due Tuesday, September 26, 2017

This assignment is due at the beginning of class on the above date and must be submitted electronically via Autolab.

### 1 Verifications

Recall the  $\diamond$  connective from assignment 2:

$$\begin{array}{c}
 \overline{A \text{ true}}^u \\
 \vdots \\
 \overline{B \text{ true}} \\
 \hline
 \diamond(A, B, C) \text{ true} \quad \diamond I_1^u
 \end{array}
 \qquad
 \begin{array}{c}
 \overline{A \text{ true}}^u \\
 \vdots \\
 \overline{C \text{ true}} \\
 \hline
 \diamond(A, B, C) \text{ true} \quad \diamond I_2^u
 \end{array}$$

$$\begin{array}{c}
 \overline{B \text{ true}}^u \quad \overline{C \text{ true}}^v \\
 \vdots \qquad \qquad \vdots \\
 \overline{D \text{ true}} \quad \overline{D \text{ true}} \\
 \hline
 \diamond(A, B, C) \text{ true} \quad A \text{ true} \quad D \text{ true} \quad D \text{ true} \quad \diamond E^{u,v} \\
 \hline
 D \text{ true}
 \end{array}$$

**Task 1** (9 points). Give rules for forming the judgments that  $\diamond(A, B, C)$  has a verification and that  $\diamond(A, B, C)$  can be used.

**Task 2** (5 points). On assignment 1, you showed

$$(\neg A \wedge B) \supset ((A \supset B) \supset (\neg A \supset \neg B)) \supset \perp \text{ true}.$$

Give a verification for this proposition.

**Task 3** (14 points). For each of the following propositions, give a verification and its corresponding proof term:

- a.  $\perp \supset \top$
- b.  $(A \supset B) \supset (B \supset C) \supset (A \supset C)$

## 2 Quantification

It is important to note that quantification extends as far to the right as syntactically possible. For example, the proposition  $\exists x : \tau.A(x) \supset \forall x : \tau.A(x)$  should be interpreted as  $\exists x : \tau.(A(x) \supset \forall x : \tau.A(x))$  and not as  $(\exists x : \tau.A(x)) \supset (\forall x : \tau.A(x))$ . Tutch implements the same convention.

### 2.1 Distributivity properties

In class, we saw that universal quantification distributes over conjunction, that is,

$$(\forall x : \tau.A(x) \wedge B(x)) \equiv (\forall x : \tau.A(x)) \wedge \forall x : \tau.B(x) \text{ true.}$$

In this section, we will explore various other distributivity properties.

**Task 4** (10 points). Dually, existential quantification distributes over disjunction, that is,

$$(\exists x : \tau.A(x) \vee B(x)) \equiv (\exists x : \tau.A(x)) \vee \exists x : \tau.B(x) \text{ true.}$$

In this task, you will show this equivalence by giving a natural deduction proof of each of the following directions:

- a.  $(\exists x : \tau.A(x) \vee B(x)) \supset (\exists x : \tau.A(x)) \vee \exists x : \tau.B(x) \text{ true}$
- b.  $(\exists x : \tau.A(x)) \vee (\exists x : \tau.B(x)) \supset \exists x : \tau.A(x) \vee B(x) \text{ true}$

### 2.2 Constructive and classical quantification

**Task 5** (9 points). For each of the following judgments, give a constructive natural deduction proof and the corresponding proof term if it is constructively valid. If it is not constructively valid, state this. *N.B. The following judgments are all classically valid.*

- a.  $(\neg \forall x : \tau.\neg A(x)) \supset \exists x : \tau.A(x) \text{ true}$
- b.  $(\exists x : \tau.A(x)) \supset \neg \forall x : \tau.\neg A(x) \text{ true}$

### 2.3 Tutch, Quantified

Tutch uses the concrete syntax  $?x:\tau.A(x)$  and  $!x:\tau.A(x)$  for  $\exists x : \tau.A(x)$  and  $\forall x : \tau.A(x)$ , respectively. We encourage you to review the scoping rules for quantifiers described at the beginning of Section 2 of this assignment before starting this portion of the assignment. Please see the Tutch manual for more information on how to use quantifiers in Tutch.

**Task 6** (6 points). Prove each of the following propositions using Tutch. Place the proof for part a (and only the proof for part a) in `hw3_6a.tut`, ..., and the proof for part c (and only the proof for part c) in `hw3_6c.tut`.

- a. proof apply :  $(!x:t.A(x) \Rightarrow B(x)) \Rightarrow (!x:t.A(x)) \Rightarrow (!x:t.B(x));$
- b. proof instance :  $(!x:t.A(x)) \ \& \ (?y:t.B(y)) \Rightarrow ?z:t.A(z);$
- c. proof frobenius :  $(R \ \& \ ?x:t.Q(x)) \Leftrightarrow ?x:t.(R \ \& \ Q(x));$

## Submitting your solutions

Please generate a tarball containing your solution files by running

```
$ tar cf hw3.tar hw3.pdf hw3_6a.tut hw3_6b.tut hw3_6c.tut
```

and submit the resulting `hw3.tar` file to Autolab.