1 Local Soundness and Completeness (12 pts)

See the Lecture 3 (Harmony) notes for a discussion of local soundness and completeness.

1.1 Hearts

Consider a connective defined by the following rules:

\[
\begin{array}{c}
\frac{A \text{ true}}{A \lor B \text{ true}} \lor I_L \\
\frac{B \text{ true}}{A \lor B \text{ true}} \lor I_R \\
\vdots \\
\frac{A \lor B \text{ true}}{C \text{ true}} E^{u,v}
\end{array}
\]

Task 3 (3 pts). Is this connective locally sound? If so, show the reduction; if no, explain (informally) why no such reduction exists.

Solution. It is not sound. The setup for one case that we must consider is as follows:

\[
\begin{array}{c}
\frac{D}{A \lor B \text{ true}} \lor I_L \\
\frac{A \text{ true} \quad B \text{ true}}{A \lor B \text{ true}} \lor I_R \\
\vdots \\
\frac{A \lor B \text{ true}}{C \text{ true}} E^{u,v}
\end{array}
\]

We need a proof of \( C \). Substituting \( D \) for the assumption \( u \) leaves a derivation

\[
\begin{array}{c}
\frac{\text{true}}{B \text{ true}} v \\
\vdots \\
\frac{\text{true}}{C \text{ true}}
\end{array}
\]

but we do not have a proof of \( B \) to substitute for the remaining hypothesis. The other case, where the introduction rule used is \( \lor I_R \), is unsound for symmetric reasons.

It should not be surprising that this connective is unsound, since it mixes the intro rules for \( \lor \) with an elimination rule for \( \land \): the elim rule takes out more than the intro rule puts in.
Task 4 (3 pts). Is this connective locally complete? If so, show the expansion; if not, explain (informally) why no such expansion exists.

Solution. The connective is locally complete. Here is one possible expansion:

\[ D \quad A \quad ♥ \quad B \quad true \implies E \]

\[ D \quad A \quad ♥ \quad B \quad true \]

\[ D \quad A \quad ♥ \quad B \quad true \quad \implies w \]

\[ D \quad A \quad ♥ \quad B \quad true \quad \implies w \quad E \quad u,v \]

It is also possible to use ♥IR and the assumption v of B true. The fact that there are two possible expansions should make you suspicious about soundness: not all of the information that the elim rule produces is necessary to introduce the connective.

1.2 Clubs

Consider a connective defined by the following rules:

\[ \begin{align*}
A \text{ true } & \overset{u}{\Rightarrow} D \\
B \text{ true } & \overset{u}{\Rightarrow} ♥I_L \\
(C \text{ true } & \overset{u}{\Rightarrow} ♥I_R) \\
(A, B, C) \text{ true } & \overset{u}{\Rightarrow} ♥E^{u,v}
\end{align*} \]

Task 1 (3 pts). Is this connective locally sound? If so, show the reduction; if not, explain (informally) why no such reduction exists.

Solution. Yes, it is locally sound. Two reductions are necessary:

\[ \begin{align*}
A \text{ true } & \overset{w}{\Rightarrow} D \\
B \text{ true } & \overset{w}{\Rightarrow} ♥I_L \\
(C \text{ true } & \overset{w}{\Rightarrow} ♥I_R) \\
(A, B, C) \text{ true } & \overset{w}{\Rightarrow} ♥E^{u,v}
\end{align*} \]

\[ \begin{align*}
A \text{ true } & \overset{w}{\Rightarrow} E \\
B \text{ true } & \overset{w}{\Rightarrow} F_1 \\
(C \text{ true } & \overset{w}{\Rightarrow} F_2) \\
(A, B, C) \text{ true } & \overset{w}{\Rightarrow} ♥E^{u,v}
\end{align*} \]

i.e. \([E/w][D/u]F_1\)

\[ \begin{align*}
A \text{ true } & \overset{w}{\Rightarrow} D \\
C \text{ true } & \overset{w}{\Rightarrow} ♥I_R \\
(A, B, C) \text{ true } & \overset{w}{\Rightarrow} ♥E^{u,v}
\end{align*} \]

\[ \begin{align*}
A \text{ true } & \overset{w}{\Rightarrow} E \\
B \text{ true } & \overset{w}{\Rightarrow} F_1 \\
C \text{ true } & \overset{w}{\Rightarrow} F_2 \\
(A, B, C) \text{ true } & \overset{w}{\Rightarrow} ♥E^{u,v}
\end{align*} \]

i.e. \([E/w][D/u]F_2\)

Task 2 (3 pts). Is this connective locally complete? If so, show the expansion; if not, explain (informally) why no such expansion exists.
**Solution.** This connective is not locally complete.

Note that the intro rules are essentially the intros for \((A \supset B) \lor (A \supset C)\), but the elim rule is essentially the elim rule for \(A \supset (B \lor C)\). The problem is that the intro rule forces the choice between \(B\) and \(C\) to be made too early for this elimination rule. For example, if we try to expand using \(\la I_L\) first, then we get stuck in the second branch of the elim:

\[
\begin{array}{c}
\frac{D}{(A, B, C) \text{ true}}
\frac{A \text{ true} \quad B \text{ true}}{E_{v_1, v_2}^u}
\frac{\la I_L}{(A, B, C) \text{ true}}
\end{array}
\]

Symmetrically, if we tried \(\la I_R\) first, then we’d get stuck in the first branch.

Finally, we cannot use \(\la E\) at the outside (like you do for disjunction) because then we don’t have a proof of \(A\):