Adversarially Robust Streaming Algorithms

Classic Streaming Algorithms

Modeled by updates to a large vector $x \in \mathbb{R}^n$

- At time $t = 1,2, \ldots$, receive update (i_t, Δ_t), causing change

$$x_{i_t} \leftarrow x_{i_t} + \Delta_t$$

- If for all t, $\Delta_t \geq 0$, the stream is called insertion-only

- At any point, algorithm computes function $f(x)$ using small space

- E.g. $f(x) = F_2 = |x|_2^2 = \sum_i x_i^2$ or $f(x) = F_0 = |\{i \text{ such that } x_i \neq 0\}|$
- Algorithm stores a small sketch $S(x)$ of the data, much smaller than n bits

Example: $\mathbf{F_2}$ – Estimation

- Output $\widetilde{F_2}$ for which $(1-\epsilon)\cdot F_2 \le \widetilde{F_2} \le (1+\epsilon)\cdot F_2$ (recall $F_2 = \sum_i x_i^2$)
- Choose a random matrix $S \in \{-\epsilon, \epsilon\}^{\frac{1}{\epsilon^2} \times n}$
  - Entries can be 4-wise independent
- Maintain $S \cdot x$ in the stream
  - Given an update $x_{i_t} \leftarrow x_{i_t} + \Delta_t$, set $S \cdot x \leftarrow S \cdot x + \Delta_t \cdot S_{*,\, i_t}$
- Use $|S \cdot x|_2^2$ to estimate $|x|_2^2$
  - $E_S\left[|S \cdot x|_2^2\right] = |x|_2^2$ and $\mathrm{Var}_S\left[|S \cdot x|_2^2\right] = O(\epsilon^2 |x|_2^4)$
- $O(\frac{\log n}{\epsilon^2})$ bits of memory

$$S = \epsilon \cdot \begin{bmatrix} -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Example: $\mathbf{F_0}$ – Estimation in Insertion Streams

- Output $\widetilde{F_0}$ with $(1-\epsilon)\cdot F_0 \le \widetilde{F_0} \le (1+\epsilon)\cdot F_0$ (recall $F_0 = |\{i \text{ with } x_i \neq 0\}|$)

- Choose a hash function $h\colon \{1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, M\}$, where $M = O\left(n^2\right)$
  - With good probability, no collisions

- Maintain the smallest $t = \frac{100}{\epsilon^2}$ hash values in the stream

- Output Z = tM/v, where v is the t-th smallest hash value
  - Smallest hash value about $\frac{M}{F_0}$, so v should be about $\frac{tM}{F_0}$

- $O(\epsilon^{-2} \cdot \log n)$ bits of memory. Can improve to $O(\epsilon^{-2} + \log n)$ bits

Tracking Algorithms

- $x^{(t)} :=$ the stream vector after updates $1,2, \ldots t$
- Algorithm must output $R^{(t)}$ so

$$R^{(t)} = (1 \pm \epsilon)f(x^{(t)})$$

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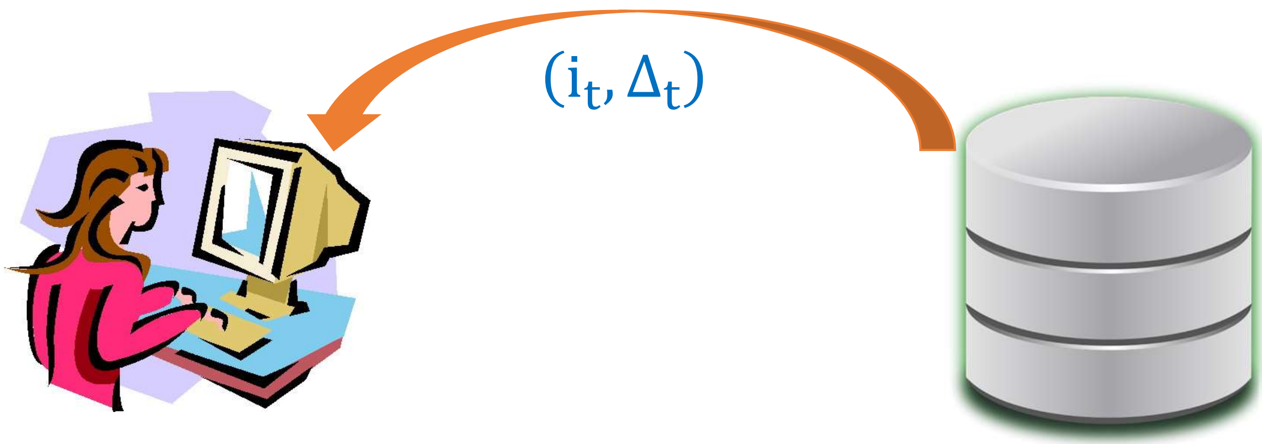

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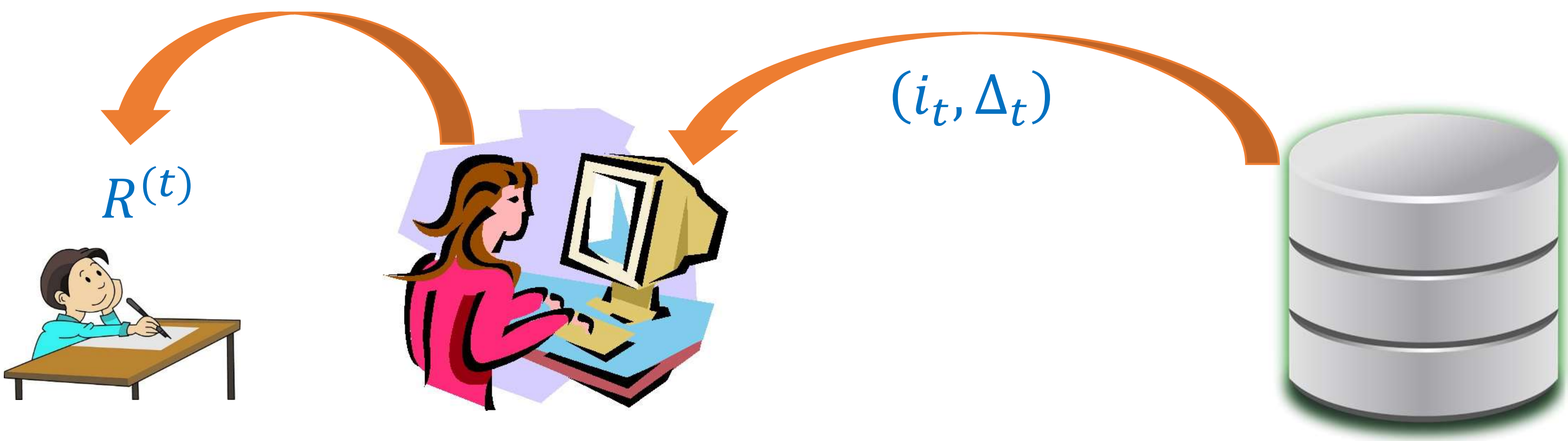

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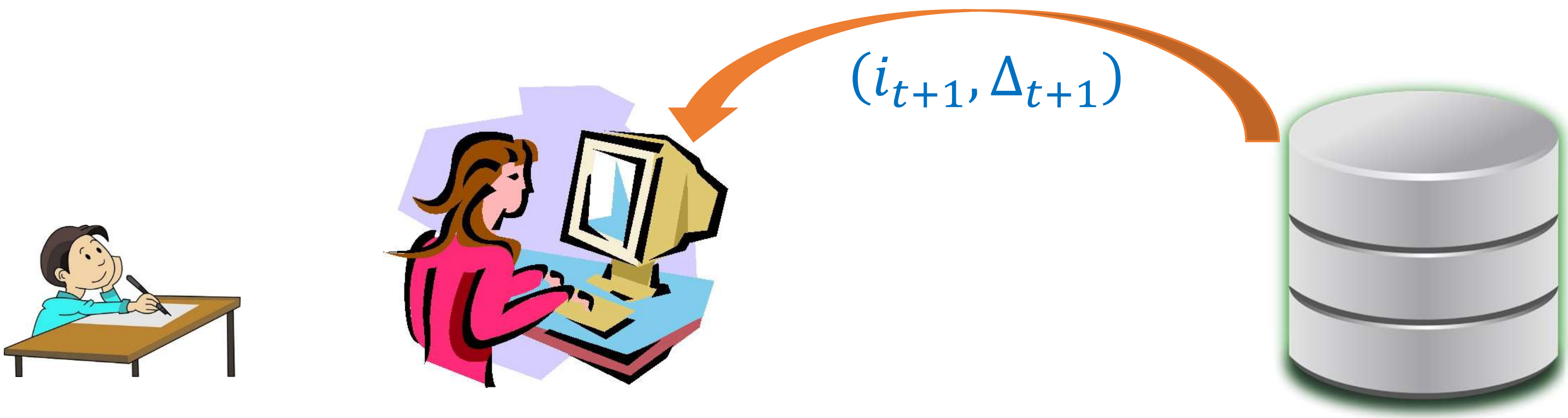

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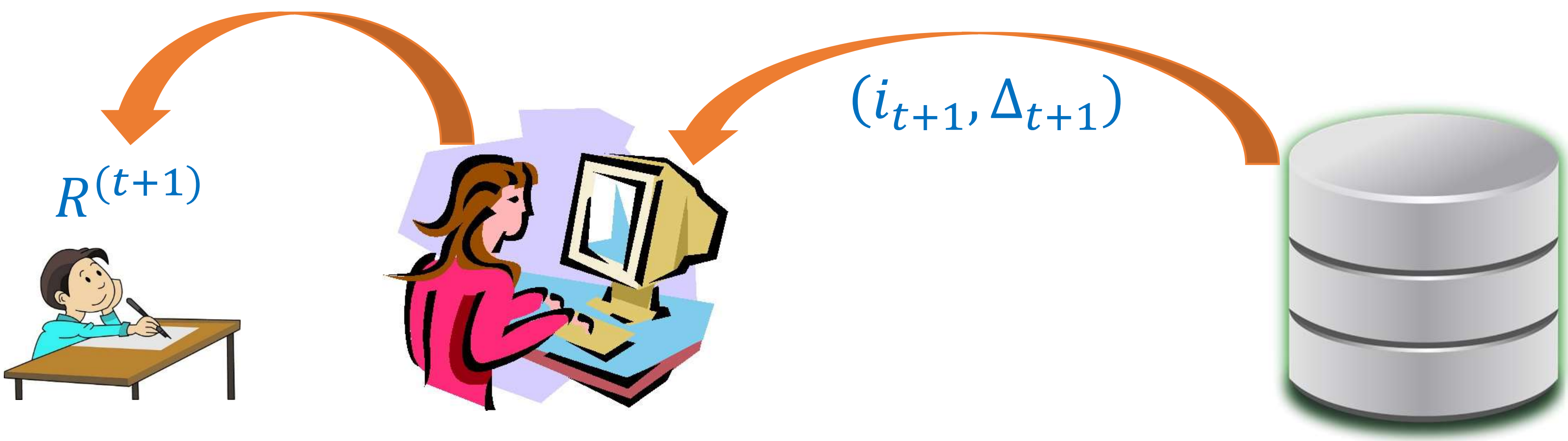


Adversarial Streams

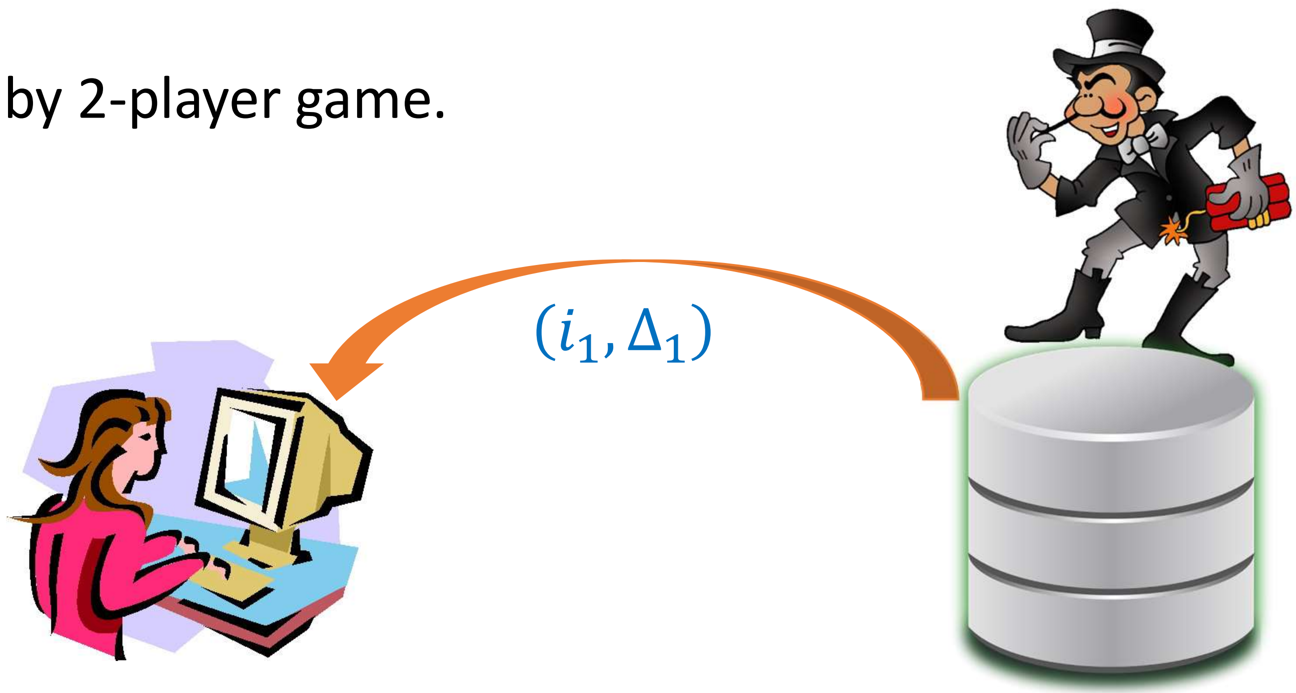
- Classic streams: Data is fixed before algorithm starts: future data does not depend on outputs $\mathrm{R}^{(t)}$!

  - Future data often depends on past decisions: no known guarantees for streaming algorithms in this case!

- Adversarial Streams: Adversary controls stream updates: sees $\mathrm{R}^{(t)}$, then gets to choose $(\mathrm{i}_{t+1}, \Delta_{t+1})$.

Adversarial Streams

Modeled by 2-player game.



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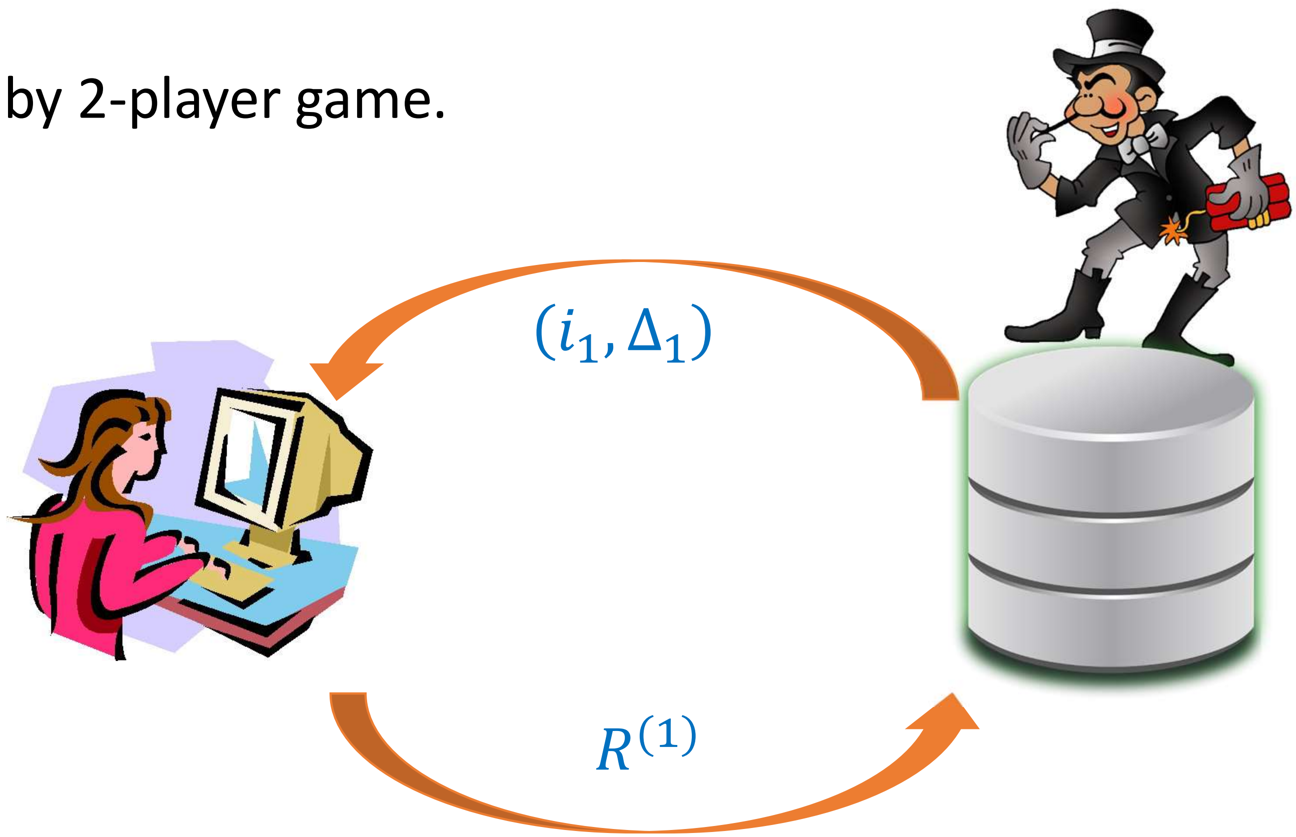

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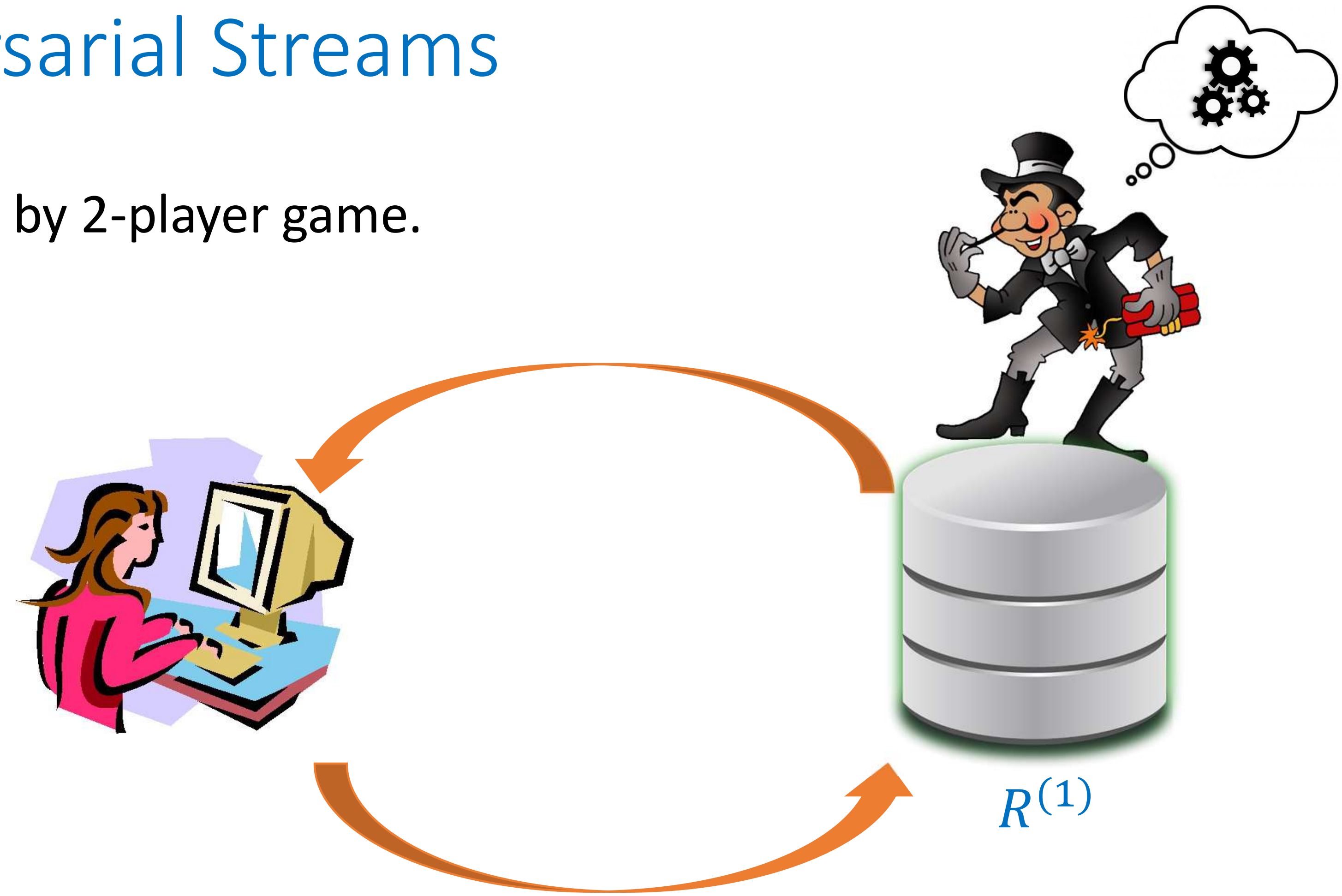

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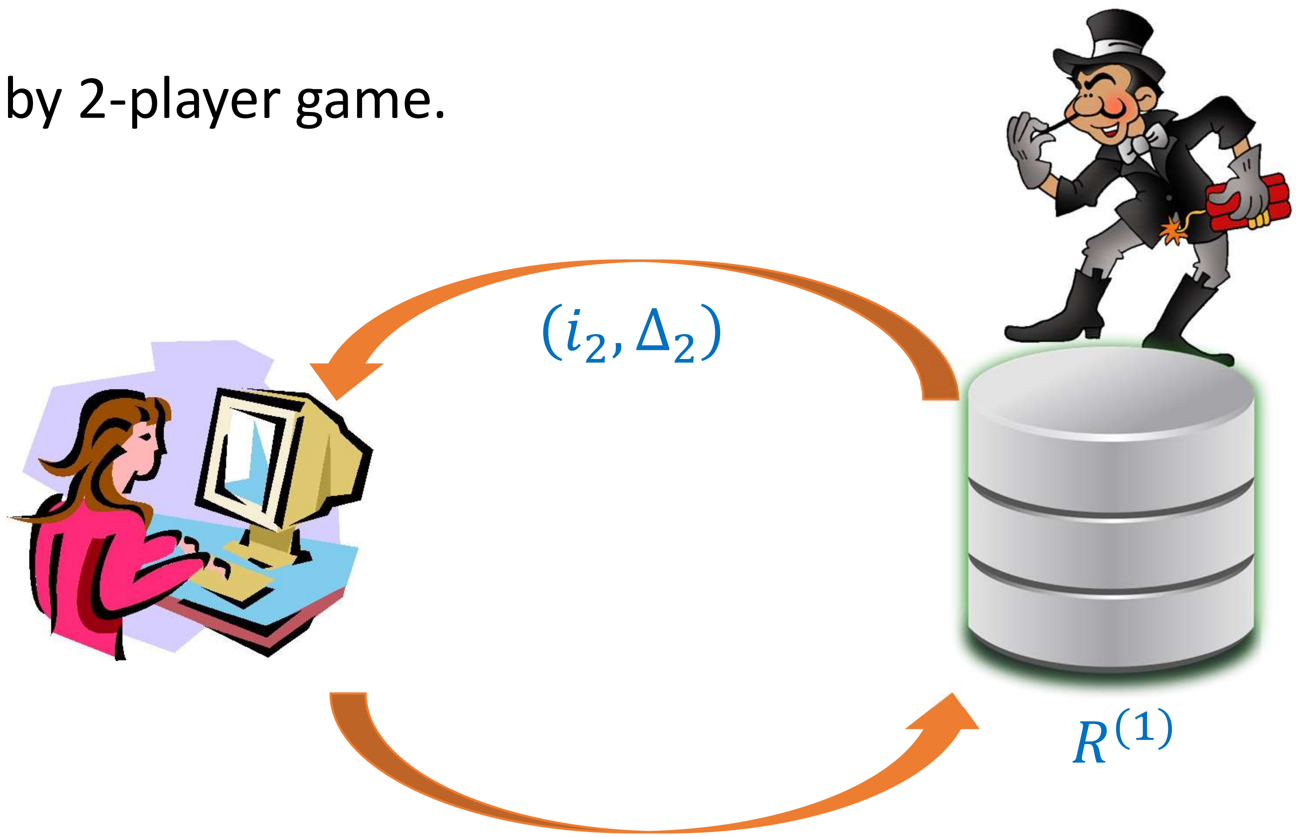

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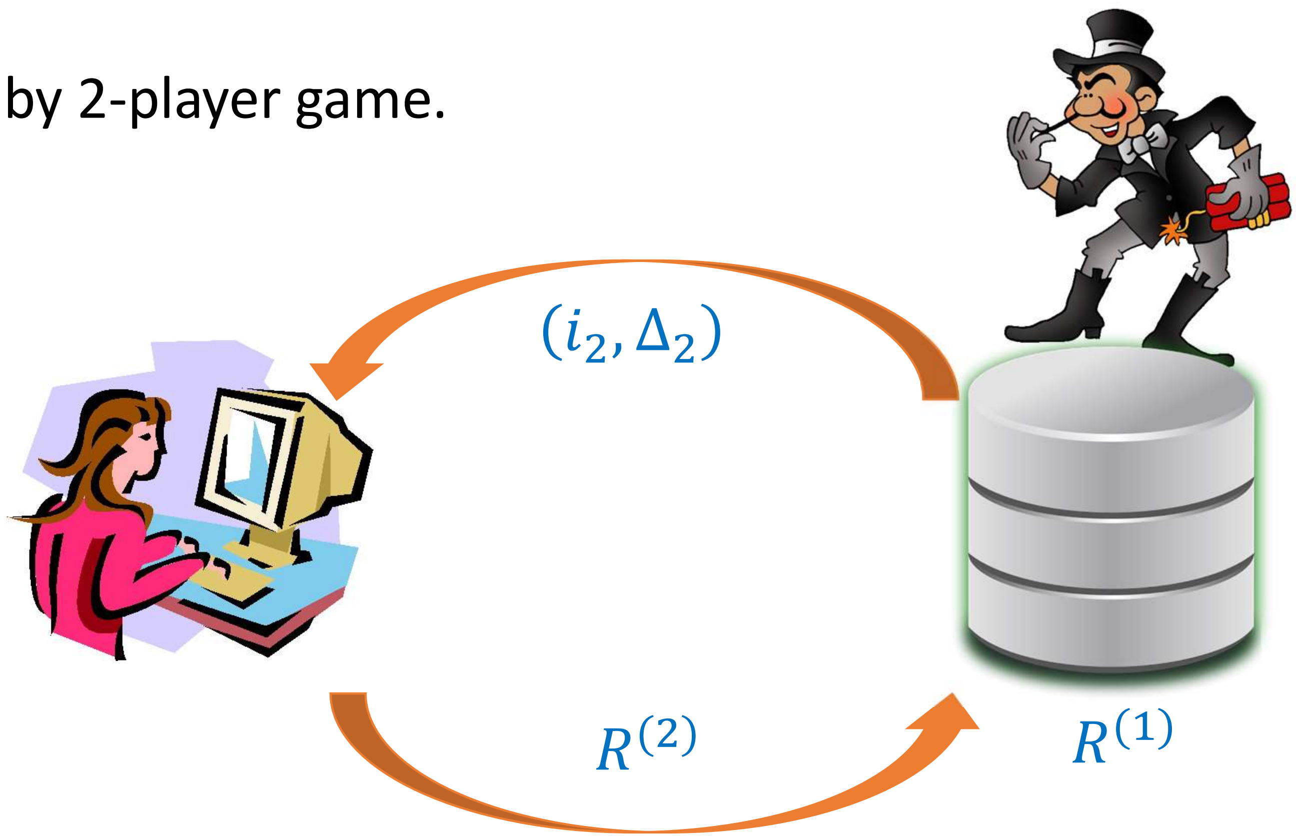

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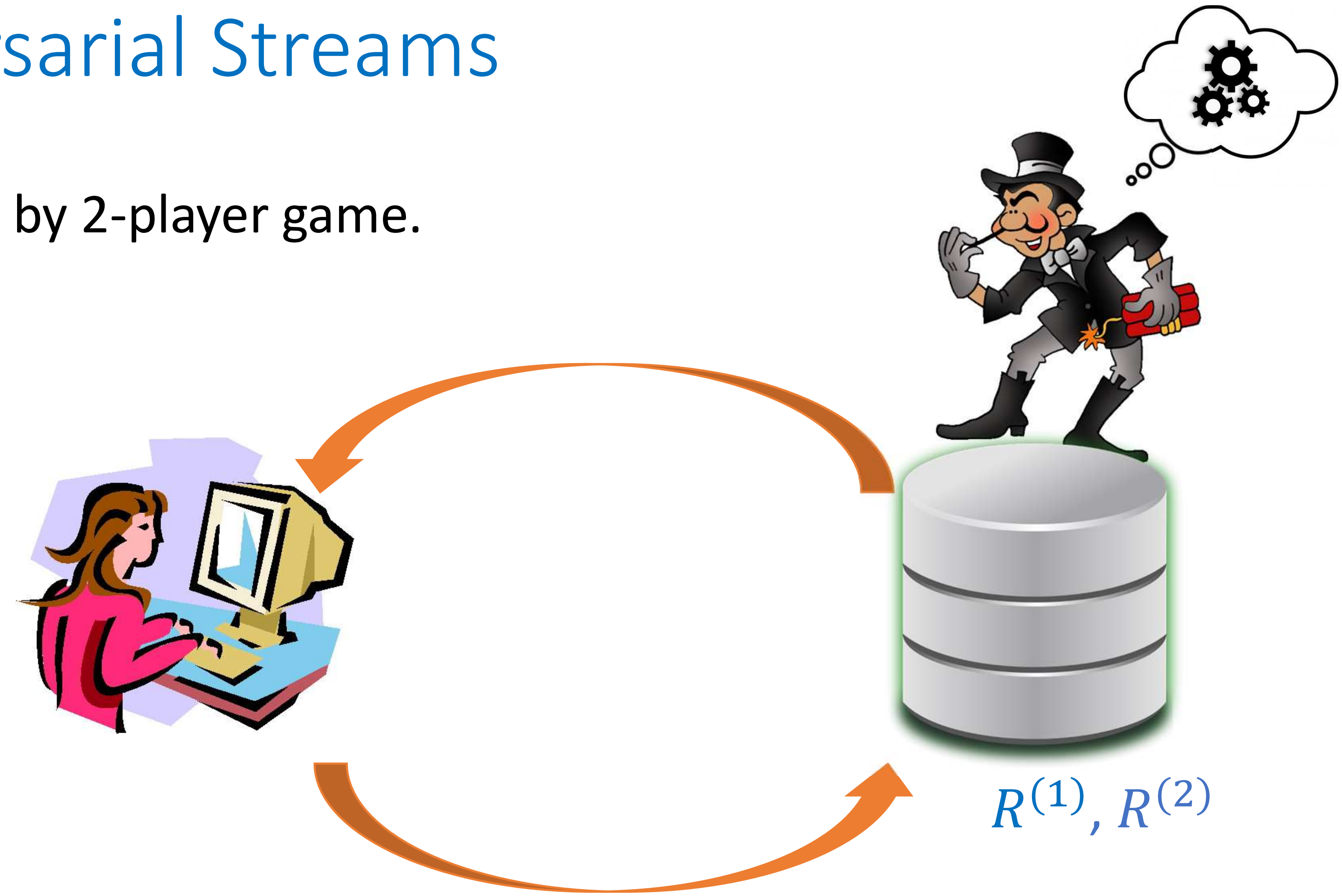

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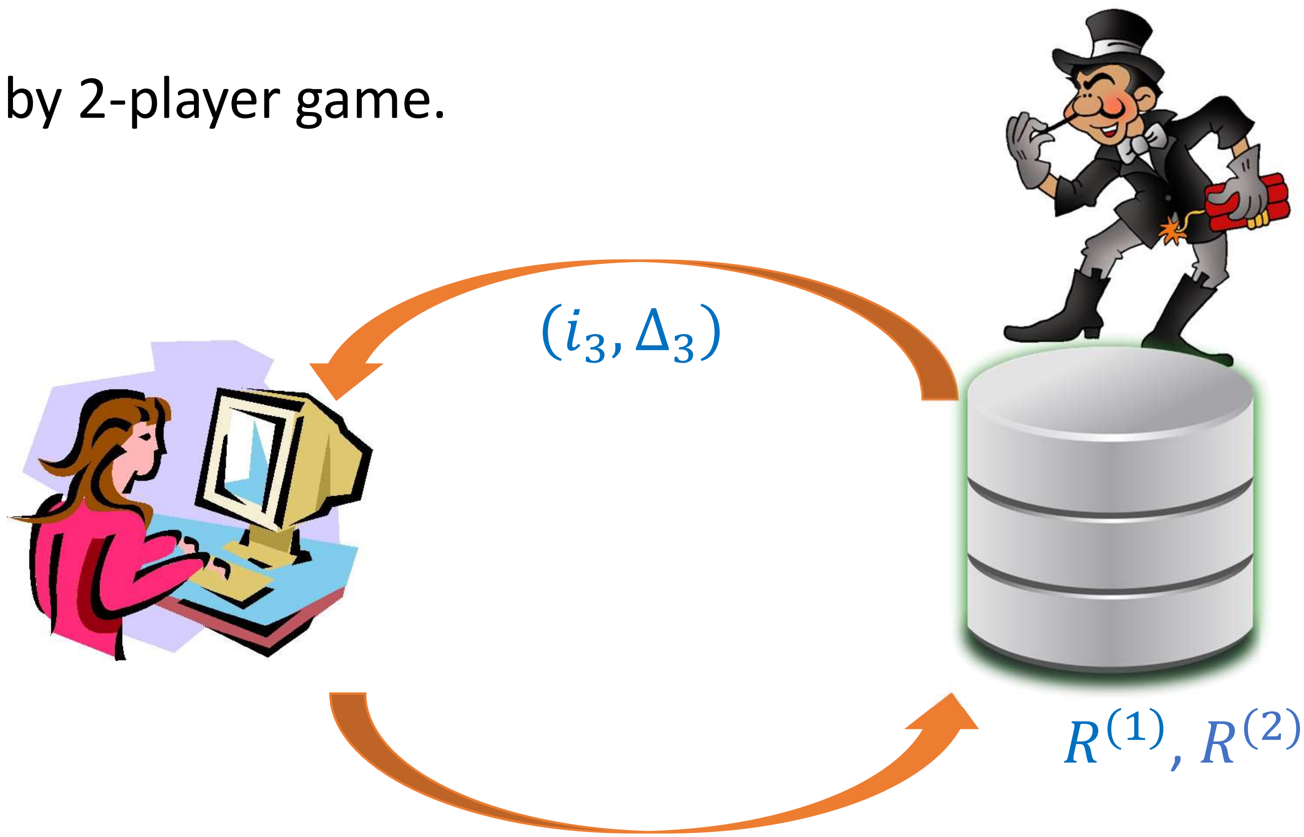

Adversarial Streams

Goal of Adversary: Make Algorithm fail to output an ϵ-approximation:

- Adversary wants: $R^{(t)} \neq (1 \pm \epsilon)f(x^{(t)})$ at some time t
- Adversary has unbounded computational power, knows entire history of outputs $R^{(1)}, R^{(2)}, \ldots R^{(t-1)}$ at time t
- Deterministic algorithms are adversarially robust, however most streaming algorithms provably **must** be randomized

Classic Streaming Algorithms Not Robust!

Theorem: the classic **AMS Sketch** (Alon, Matias, Szegedy '96) for estimating F_2 is not adversarially robust!

- Even in insertion only streams, meaning $\Delta_t \geq 0$ for all t.

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- Even in insertion only streams, meaning $\Delta_t \geq 0$ for all t.

We need new algorithms!

Generic Transformations [Ben-Eliezer, Jayaram, W, Yogev]

Give two **generic** methods to **transform** any streaming algorithm $\mathcal{A}$ into an *adversarially robust* algorithm $\mathcal{A}'$, with a mild space overhead:

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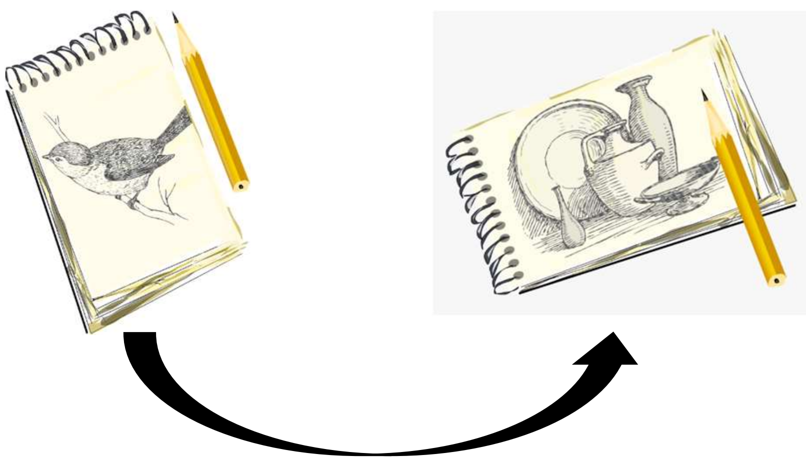
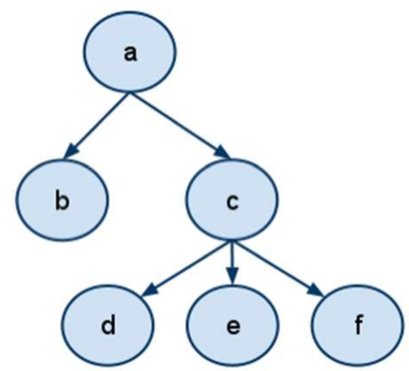
Sketch Switching



Computation Paths



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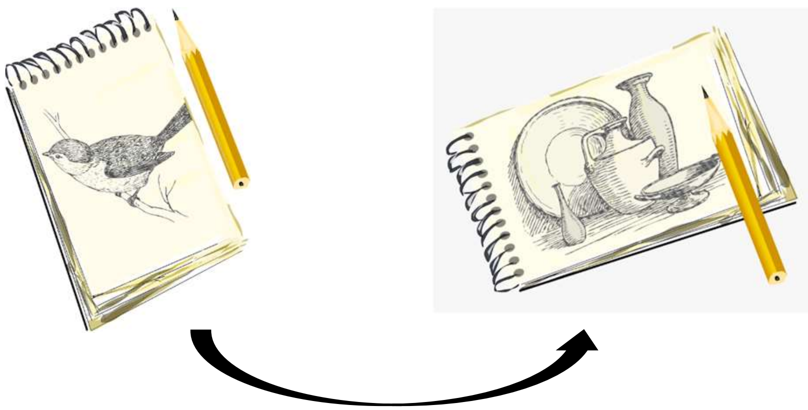
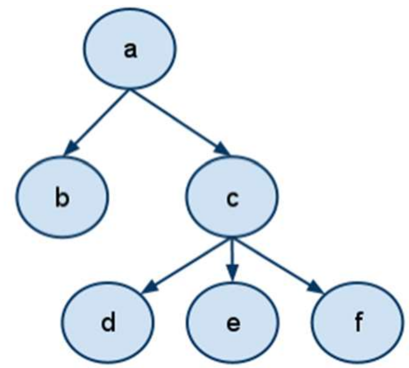


Useful for exploiting algorithms $\mathcal{A}$ which provide tracking better than one shot + a union bound over all time steps t.

Computation Paths



Useful for exploiting algorithms $\mathcal{A}$ with better dependence on failure probability δ than multiplicative $O(\log 1/\delta)$.

Flip Number

Definition (informal): For a function $f: \mathbb{R}^n \to \mathbb{R}$, define the ϵ*-flip number* $\lambda_\epsilon(f)$ to be the maximum number of times $f(x^{(t)})$ can change by a factor of $(1+\epsilon)$ after $\text{poly}(n)$ updates.

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Example: $\mathrm{f(x)} = \|\mathrm{x}\|_\mathrm{p}^\mathrm{p} = \sum_\mathrm{i} |\mathrm{x_i}|^\mathrm{p}$**,** then for insertion only streams

1. $\|\mathrm{x}^{(0)}\|_\mathrm{p}^\mathrm{p} = 0$
2. $\|\mathrm{x}^{(\mathrm{poly(n)})}\|_\mathrm{p}^\mathrm{p} \le \mathrm{poly(n)}$

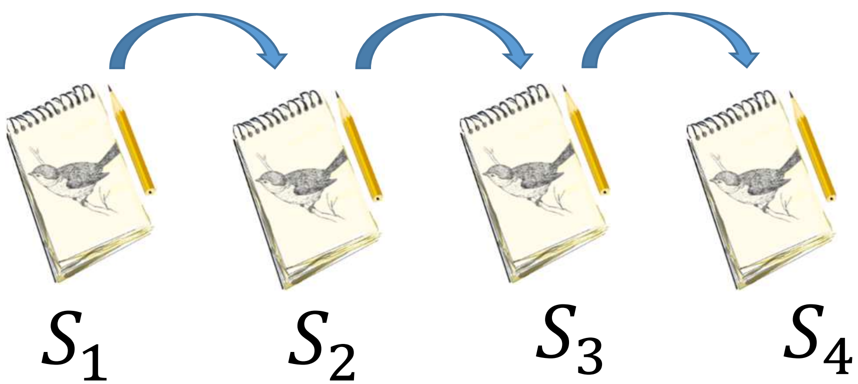
So $\lambda_\epsilon(\mathrm{f}) = \log_{(1+\epsilon)}(\mathrm{poly(n)}) = O\left(\frac{1}{\epsilon}\log \mathrm{n}\right)$

Sketch Switching

1. Keep multiple ($\lambda_\epsilon(f)$ many) independent sketches concurrently.

2. Only use output of one sketch S_i at a time.

3. Once estimate $R^{(t)}$ of S_i changes by $(1+\epsilon)$, info about S_i is leaked, throw S_i away!



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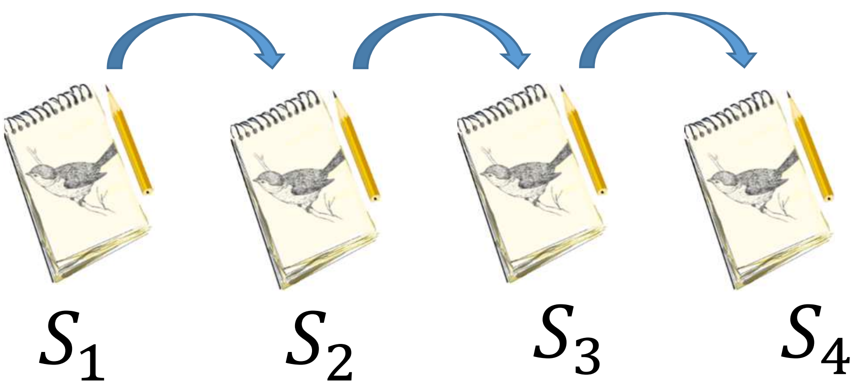
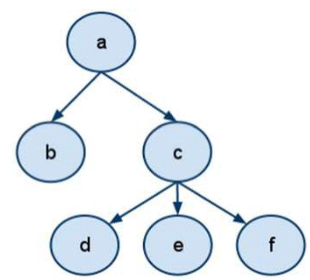


Computation Paths

Streaming algorithms can be made robust by setting failure probability δ small enough:

$$\delta' = \delta \cdot n^{-O(\lambda_\epsilon(f))}$$

1. Only need to change output $\lambda_\epsilon(f)$ times!
2. Stream is length $\mathrm{poly}(n)$, and output is O(log n) bits, so only $n^{O(\lambda_\epsilon(f))}$ possible "computation paths" between algorithm and adversary.
3. Setting δ' small enough, can union bound over all of them!



Theorem (informal)**:** For $f: \mathbb{R}^n \to \mathbb{R}$, let $\mathcal{A}$ be any algorithm which ϵ-tracks $f\left(x^{(t)}\right)$. Then there is an adversarially robust algorithm $\mathcal{A}'$ for ϵ-tracking $f\left(x^{(t)}\right)$ using space $O\left(\lambda_\epsilon(f) \cdot \mathrm{Space}(\mathcal{A})\right)$.

Theorem (informal)**:** Let $\mathcal{A}$ be any algorithm which ϵ-tracks $f\left(x^{(t)}\right)$ with probability $1 - \delta$ using space $\mathrm{Space}(\mathcal{A}, \delta)$. Then there is an adversarially robust algorithm $\mathcal{A}'$ that uses $\mathrm{Space}(\mathcal{A}, \delta')$ where

$$\delta' = \delta \cdot n^{-O(\lambda_\epsilon(f))}$$

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Theorem (informal)**:** For $f: \mathbb{R}^n \to \mathbb{R}$, let $\mathcal{A}$ be any algorithm which ϵ-tracks $f\left(x^{(t)}\right)$**.** Then there is an adversarially robust algorithm $\mathcal{A}'$ for ϵ-tracking $f\left(x^{(t)}\right)$ using space $O\left(\lambda_\epsilon(f) \cdot \mathrm{Space}(\mathcal{A})\right)$.

Application of Sketch Switching

Theorem (Blasiok SODA '18): There is a streaming algorithm that ϵ-tracks the number of **distinct elements** in an insertion only stream, defined by $\|x^{(t)}\|_0 = |\{ i : x_i \neq 0 \}|$, using space $O\left(\frac{\log \mathrm{lo}}{\epsilon^2} + \log n\right)$.

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Theorem: There is an adversarially robust streaming algorithm for ϵ-tracking the number of **distinct elements** using space:

$$O\left(\frac{\log n}{\epsilon}\left(\frac{\log\log n}{\epsilon^2} + \log n\right)\right)^*$$

In this case can improve the leading log n factor to $\log(\frac{1}{\epsilon})$

Sketch Switching

High level idea:

- Adversary wants to learn about your sketch and randomness to fool it
- We carefully reveal information about our sketches
- As soon as we reveal any new information, immediately make this information irrelevant

Sketch Switching

Sketch Switching:

1. Create $O(\lambda_{\frac{\epsilon}{12}}(f))$ independent sketches $S_1, \ldots, S_k$, each providing a $(1 \pm \frac{\epsilon}{10})$-approximation. Set $i = 1$ and $R^{(0)} = 0$

2. At time t, if $\text{Estimate}\ (S_i) \notin \left(1 \pm \frac{\epsilon}{3}\right) R^{(t-1)}$
   1. Set $R^{(t)} = \text{Estimate}(S_i)$ and throw out S_i
   2. $i \leftarrow i + 1$

   Otherwise set $R^{(t)} = R^{(t-1)}$

Sketch Switching Proof

- Can assume adversary is deterministic by averaging
- This fixes the part of the stream the adversary gives S^i after S^{i-1} returned an answer Out
  - The stream does not depend on S^i though it may depend on $S^1, S^2, \ldots, S^{i-1}$
  - So, S^i is correct at all positions in new stream
- S^i outputs old value Out until its value Out' $\notin [\left(1-\frac{\epsilon}{3}\right)\text{Out}, \left(1+\frac{\epsilon}{3}\right)\text{Out}]$, at which point we switch to S^{i+1}
- Might worry you learn something about S^i until it outputs Out', and you do, but S^i is correct on whatever fixed stream you choose until S^i outputs Out'

Sketch Switching Proof

- If S^i provides a $(1 \pm \frac{\epsilon}{10})$-approximation, then if Out' $\notin [\left(1 - \frac{\epsilon}{3}\right) \text{Out}, \left(1 + \frac{\epsilon}{3}\right) \text{Out}]$, necessarily f has changed by a $(1 \pm \frac{\epsilon}{12})$ factor
  - Number of sketches we need is bounded by $\lambda_{\frac{\epsilon}{12}}(f)$
- Conversely, if f changes by a $(1 \pm \epsilon)$ factor, necessarily Out' $\notin [\left(1 - \frac{\epsilon}{3}\right) \text{Out}, \left(1 + \frac{\epsilon}{3}\right) \text{Out}]$
  - So we are correct at all times

Computation Paths

Theorem (informal)**:** For $f: \mathbb{R}^n \to \mathbb{R}$, let $\mathcal{A}$ be any algorithm which ϵ-tracks $f\left(x^{(t)}\right)$ with probability $1-\delta$ using space $L(\epsilon, \delta)$. Then there is a *robust* algorithm $\mathcal{A}'$ for using space $L(\epsilon/10, \delta')$, where

$$\delta' = \delta \cdot n^{-O(\lambda_{\epsilon/12}(f))}$$

- Streaming algorithms can be made robust by setting failure probability δ to be small!

Computation Paths: High Level Proof

Streaming algorithms can be made robust by setting failure probability to be

$$\delta' = \delta \cdot n^{-O(\lambda_{\epsilon/12}(f))}$$

1. Only need to change the output $\lambda_{\epsilon/12}(f)$ times
2. Stream is length poly(n), and output is O(log n) bits, so $n^{O(\lambda_{\epsilon/12}(f))}$ possible streams a deterministic adversary can create
3. Setting δ' small enough, can union bound over all of them

Results of [BJWY], instantiating $\lambda_\epsilon(f) = O(\epsilon^{-1}\log n)$

Problem	Non-Adversarial	Adversarial ([BJWY])
Distinct Elements (F_0)	$\tilde{O}(\epsilon^{-2} + \log n)$	$\tilde{O}(\epsilon^{-3} + \log n)$
F_p estimation, $p \in (0,2] \setminus \{1\}$	$\tilde{O}(\epsilon^{-2}\log n)$	$\tilde{O}(\epsilon^{-3}\log n)$
F_p estimation, $p > 2$	$O\left(n^{1-\frac{2}{p}}\left(\epsilon^{-3}\log^2 n + \epsilon^{-\frac{6}{p}}\log^{\frac{4}{p}+1} n\right)\right)$	Same when $\delta = O(n^{-\frac{\log}{\epsilon}})$
Heavy Hitters	$O(\epsilon^{-2}\log^2 n)$	$\tilde{O}(\epsilon^{-3}\log^2 n)$
Entropy Estimation	$O(\epsilon^{-2}\log^3 n)$	$O(\epsilon^{-5}\log^6 n)$

For adversaries with **bounded computation + Cryptographic Assumptions,** can improve some of above:

Problem	Adversarial ([BJWY])
Distinct Elements (F_0)	$\tilde{O}(\epsilon^{-2} + \log n)$ optimal even with no adversary
Entropy Estimation	$O(\epsilon^{-5}\log^4 n)$

Polynomially Bounded Adversaries

- Recall non-robust $\mathrm{F_0}$-estimation algorithm:
  - Choose a hash function $\mathrm{h}: \{1, 2, \ldots, \mathrm{n}\} \rightarrow \{0, 1, 2, \ldots, \mathrm{M}\}$, where $\mathrm{M} = \mathrm{O}\left(\mathrm{n}^2\right)$
  - Maintain smallest $\mathrm{t} = \frac{100}{\epsilon^2}$ values h(i) found when processing stream
- State of the algorithm is exactly the same if you insert the same item twice
  - Breaking this algorithm requires breaking the hash function h
- Assumption: for any c > 0 there is a d > 0 and a family of $\mathrm{n^d}$ hash functions that can be evaluated in O(log n) memory such that any $\mathrm{n^c}$-time Adversary cannot break this
  - Exponentially secure pseudorandom function (in practice, AES or SHA256)

Improvements

1. [Hassidim, Kaplan, Mansour, Matias, Stemmer]
    1. Use differential privacy
    2. Improve the [BJWY] bounds of $\widetilde{O}(\epsilon^{-3} \log n)$ to $\widetilde{O}(\epsilon^{-2.5} \log^4 n)$ for F_0, F_2, and many other streaming tasks
2. [W, Zhou]
    1. Introduce "Difference Estimators"
    2. Improve the [BJWY] bounds of $\widetilde{O}(\epsilon^{-3} \log n)$ and the $\widetilde{O}(\epsilon^{-2.5} \log^4 n)$ bounds above to $\widetilde{O}(\epsilon^{-2} \log n)$ for F_0, F_2, and many other streaming tasks
    3. Non-robust algorithms for these problems require $\Omega(\epsilon^{-2})$ bits, so our memory is optimal in ϵ (and often matches non-robust log n factors)

Difference Estimators

- Do we really need to switch our sketch whenever the output changes by $1+\epsilon$?
- Maybe? Unclear what the adversary is learning.
- If the last output Out was a $(1 \pm \epsilon)$-approximation to function value f(x), and f(x) changes to f(x‘) with $f(x') \in \left(1 \pm O(\epsilon)\right)f(x)$, do we need a brand new $1 \pm \epsilon$ approximation to f(x‘)?
- Seems wasteful. We've fixed Out - maybe we can use Out for something?
- Difference Estimator: approximate f(x‘)-f(x) up to an O(1)-factor, and add it to Out!

Difference Estimators

- Need to approximate f(x')-f(x) up to additive error $\epsilon\, f(x)$ given that f(x')-f(x) $= O(\epsilon) f(x)$
  - Can't afford to approximate each of f(x') and f(x) up to relative $1 \pm \epsilon$
  - Approximating each of f(x') and f(x) up to relative O(1) error won't give $O(\epsilon) f(x)$ additive error
- Design the first difference estimators for streams!
  - Example: $|x'|_2^2 - |x|_2^2 = |x' - x|_2^2 + 2 < x' - x, x >$
  - Approximate terms up to $O(\epsilon)|x|_2^2$ error – uses $|x' - x|_2^2 = O\left(\epsilon |x|_2^2\right)$
  - Only need $1/\epsilon$ memory to do this

Sketch-Stitching and Granularity Changing

- Suppose x is the current underlying vector
- If x grows to x' with f(x') $\geq 2f(x)$, x must first grow to x^1 with f(x^1)$= (1+\epsilon)f(x)$
- Approximate the difference f(x^1)$-f(x)$ up to C/$\log(\frac{1}{\epsilon})$-relative error for constant C > 0
- Then x must grow to a vector x^2 where f(x^2)$= (1+2\epsilon)f(x)$
  - Approximate the difference f(x^2)$-f(x)$ up to $C/(2\log\left(\frac{1}{\epsilon}\right))$-relative error
  - Important not to use [f(x^1)$-f(x)$] + [f(x^2)$-f(x_1)$] here – errors would grow too fast
- Then x must grow to a vector x^3 where f(x^3)$= (1+3\epsilon)f(x)$
  - Approximate the difference f(x^2)$-f(x)$ up to $C/(2\log\left(\frac{1}{\epsilon}\right))$-relative error
  - Approximate the difference f(x^3)$-f\left(x^2\right)$ up to C/$\log(\frac{1}{\epsilon})$-relative error
- Additive errors add to $O(\epsilon)f(x)$, using $O(\log\left(\frac{1}{\epsilon}\right))$ differences in binary representation

Achieving Adversarial Robustness

- For robustness for F_2, sketch-switch in each of $\log 1/\epsilon$ levels in a binary tree
  - Top level uses memory $\frac{1}{\epsilon^2}$ but only need to sketch-switch O(log n) times
  - Bottom level uses memory $\frac{1}{\epsilon}$ but needs to sketch-switch $\frac{1}{\epsilon}$ times
  - Overall memory bound is a sum over levels

Further Work / Open Questions

- Tight bounds in terms of flip number [Kaplan,Mansour,Nissim,Stemmer]
- Improvements for small stream length [Ben-Eliezer, Eden, Onak]
- For streams with negative updates, can one prove strong lower bounds?
- Other uses of cryptography for data streams?