15-859 : Homework 3 Solutions

1 Question 1


2 Question 2

2.1 Part 1

$S$ be a Cauchy matrix used in estimating $\ell_1$-norm of a stream. As Alice and Bob both have shared randomness, they can both generate this matrix $S$. Alice computes $Sx, Sa$ and sends both $Sx, Sa$ to Bob. This requires a communication of $O(\log(n)/\varepsilon^2)$ bits. Bob then computes $Sx + Sy$ and $Sa + Sb$ and computes $Z = \text{median}(|Sx + Sy|) + \text{median}(|Sa + Sb|)$. We have from class that with probability $9/10$, the following hold simultaneously,

$$(1 - \varepsilon)\|x + y\|_1 \leq \text{median}(|S(x + y)|) \leq (1 + \varepsilon)\|x + y\|_1$$

$$(1 - \varepsilon)\|a + b\|_1 \leq \text{median}(|S(a + b)|) \leq (1 + \varepsilon)\|a + b\|_1.$$ 

Therefore $Z$ computed by Bob satisfies, with probability $\geq 9/10$, that

$$(1 - \varepsilon)(\|x + y\|_1 + \|a + b\|_1) \leq Z \leq (1 + \varepsilon)(\|x + y\|_1 + \|a + b\|_1).$$ (1)

An issue with Cauchy random variables is that they take real values and thus cannot be communicated efficiently. Note that $\varepsilon$ is assumed to be $\Omega(1/poly(n))$. Now rounding the entries of the vectors $Sa, Sx$ to nearest multiple of $1/poly(n)$ will modify the value of $Z$ computed by at most $1/poly(n)$ and therefore for all $(x, y), (a, b)$ such that $\|x + y\|_1 + \|a + b\|_1 \geq 1$, rounding of the vectors $Sx, Sa$ still means that the value $Z$ computed satisfies $\text{[boxed]}$. In the case when $x + y = 0$ and $a + b = 0$, rounding of the vectors might lead to Bob computing a different value but as rounding changes the norm by at most $1/poly(n)$, $Z$ computed is such that $Z \leq 1/poly(n)$. Then Bob can infer that $\|x + y\|_1 + \|a + b\|_1 = 0$. Note that $\|x + y\|_1 + \|a + b\|_1$ can only take integer values and thus we covered all the cases.

2.2 Part 2

We prove a lowerbound by reducing from the index problem. Alice gets a set $S \subseteq \{n\}$ and Bob gets an index $i \in \{n\}$ and wants to determine if $i \in S$ or not. Alices sets the vectors $x = a$ to be the vectors that corresponds to the set $S$. Bob sets $y$ to be a vector such that $y_i = +1$ and $y_j$ and $b$ to be a vectors such that $b_i = -1$ and $b_j = 0$ for all $j \neq i$. Then it is easy to see that

$$\|x + y\|_1 - \|a + b\|_1 = \begin{cases} 0 & \text{if } i \notin S \\ 2 & \text{if } i \notin S. \end{cases}$$
Thus, by checking the value of $Z$ computed with $\varepsilon = 1/2$, Bob can check if $i \in S$ and will be correct with probability $\geq 9/10$. But as index problem has a lower-bound of $\Omega(n)$ one-way communication complexity, we obtain that this problem also has a $\Omega(n)$ lowerbound.

3 Part 3

Let $\tilde{Z}$ be such that $(1 - \varepsilon/2)\|a + b\|_2^2 \leq \tilde{Z} \leq (1 + \varepsilon/2)\|a + b\|_2^2$. Suppose $\|a + b\|_2^2 \geq 2$. We have $\tilde{Z} \geq 2 - \varepsilon$. In this case $\tilde{Z} - 1$ satisfies

$$(1 - \varepsilon/2)\|a + b\|_2^2 - 1 \leq \tilde{Z} - 1 \leq (1 + \varepsilon/2)\|a + b\|_2^2 - 1.$$ 

We have

$$(1 + \varepsilon)(\|a + b\|_2^2 - 1) - ((1 + \varepsilon/2)\|a + b\|_2^2 - 1) = (\varepsilon/2)\|a + b\|_2^2 - \varepsilon \geq 0.$$ 

Similarly,

$$(1 - \varepsilon/2)\|a + b\|_2^2 - 1 - (1 - \varepsilon)(\|a + b\|_2^2 - 1) = \varepsilon/2\|a + b\|_2^2 - \varepsilon \geq 0.$$ 

Therefore we have

$$(1 - \varepsilon)(\|a + b\|_2^2 - 1) \leq \tilde{Z} - 1 \leq (1 + \varepsilon)(\|a + b\|_2^2 - 1)$$

if $\|a + b\|_2^2 \geq 2$. If $\|a + b\|_2^2 < 2$, there are only two possibilities as $a, b$ have integer coordinates.

- **Case 1:** $\|a + b\|_2^2 = 1$. We have $(1 - \varepsilon/2)\tilde{Z} \leq (1 + \varepsilon/2) < 2 - \varepsilon$. So if $\tilde{Z}$ computed by Bob is $< 2 - \varepsilon$ and nonzero, then we can conclude that $\|a + b\|_2^2 = 1$ and $\|a + b\|_2^2 - 1 = 0$.

- **Case 2:** $\|a + b\|_2^2 = 0$. Then $\tilde{Z}$ computed is also equal to 0 and we can conclude that $\|a + b\|_2^2 - 1 = | - 1 | = 1$.

Therefore based on value of $\tilde{Z}$, Bob can compute $1 \pm \varepsilon$ approximation to $\|a + b\|_2^2 - 1$ and such a $\tilde{Z}$ can be computed using $O(\log(n)/\varepsilon^2)$ bits of communication as seen in class. Bob can also compute $1 \pm \varepsilon$ approximation to $\|x + y\|_1$ by sketching with Cauchy matrix and a communication of $O(\log(n)/\varepsilon^2)$ bits. Now multiplying both values gives a $1 \pm O(\varepsilon)$ approximation and appropriately scaling $\varepsilon$ lets Bob compute a $Z$ such that

$$(1 - \varepsilon)(\|x + y\|_1 \cdot \|a + b\|_2^2 - 1)) \leq Z \leq (1 + \varepsilon)(\|x + y\|_1 \cdot \|a + b\|_2^2 - 1))$$

using a protocol that has communication complexity $O(\log(n)/\varepsilon^2)$. Rounding issues can be similarly solved as in Part 1.

4 Part 4

We again show a lowerbound by reducing from the index problem. Alice obtains a set $S \subseteq [n]$ and Bob obtains an index $i \in [n]$ and Bob wants to determine if $i \in S$ or not. As we’ve seen, this problem has $\Omega(n)$ lower bound on the one-way communication complexity.

Alice sets the vector $a$ to correspond to the set $S$ i.e., $a_j = 1$ if $j \in S$ and 0 otherwise and sets $x = 0$. Bob sets the vector $b = 0$ and $y$ to be the vectors such that $y_i = 1$ and $y_j = 0$ for all $j \neq i$. We now have $\|(D_a - D_b) \cdot (x - y)\|_1 = 1$ if $i \in S$ and 0 otherwise which can be distinguished using a value of $\varepsilon = 1/2$. Lowerbound on Index problem implies a $\Omega(n)$ lower bound on this problem.