Outline

- More on Streaming Lower Bounds
  - More Properties about the INDEX problem
  - Lower Bounds for p-norm estimation
- NP-Hard Variants of Low Rank Approximation
  - Non-negative Matrix Factorization
  - Weighted Low Rank Approximation

Aspects of 1-Way Communication of Index

- Alice has $x \in \{0,1\}^n$
- Bob has $i \in [n]$
- Alice sends a (randomized) message M to Bob
- $I(M ; X \mid R) = \text{sum}_i\, I(M ; X_i \mid X_{< i}, R)$

  $\geq \text{sum}_i\, I(M; X_i \mid R)$

  $= n - \text{sum}_i\, H(X_i \mid M, R)$
- Fano: $H(X_i \mid M, R) \leq H(\delta)$ if Bob can guess X_i with probability $> 1- \delta$
- $CC_\delta(\text{Index}) \geq I(M ; X \mid R) \geq n(1-H(\delta))$

The same lower bound applies if the protocol is only correct on average over x and i drawn independently from a uniform distribution

Distributional Communication Complexity

f(X,Y)?

X Y

- $(X, Y) \sim \mu$
- μ-distributional complexity $D_\mu(f)$: the minimum communication cost of a protocol which outputs f(X,Y) with probability 2/3 for $(X, Y) \sim \mu$
  - Yao's minimax principle: $\mathrm{R(f)} = \max_\mu D_\mu(\mathrm{f})$
- 1-way communication: Alice sends a single message M(X) to Bob

Indexing is Universal for Product Distributions [Kremer, Nisan, Ron]

- Communication matrix A_f of a Boolean function $f: X \times Y \to \{0,1\}$ has (x,y)-th entry equal to f(x,y)
- $\max_{\text{product } \mu} D_\mu(f) = \Theta(\text{VC} - \text{dimension})$ of A_f
- Implies a reduction from Index is optimal for product distributions

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Indexing with Low Error

- Index Problem with 1/3 error probability and 0 error probability both have $\Omega(n)$ communication

- Sometimes, want lower bounds in terms of error probability

- Indexing on Large Alphabets:
  - Alice has $x \in \{0,1\}^{n/\delta}$ with $wt(x) = n$, Bob has $i \in [n/\delta]$
  - Bob wants to decide if $x_i = 1$ with error probability δ
  - [Jayram, W] 1-way communication is $\Omega(n \log(1/\delta))$
  - Can be used to get an $\Omega(\log\left(\frac{1}{\delta}\right))$ bound for norm estimation
  - We've seen an $\Omega(\log n + \epsilon^{-2} + \log\left(\frac{1}{\delta}\right))$ lower bound for norm estimation
  - There is an $\Omega(\epsilon^{-2} \log\frac{1}{\delta} \log n)$ bit lower bound

Beyond Product Distributions

$Although\ R(f) = \max_{\mu} D_{\mu}(f)$, *it may be that* $\max_{\mu} D_{\mu}(f) \gg \max_{product\ \mu} D_{\mu}(f)$, *so one often can't get good lower bounds by looking at product distributions…*

Example: set disjointness

Non-Product Distributions

- Needed for stronger lower bounds
- Example: approximate $|x|_\infty$ up to a multiplicative factor of B in a stream
  - Lower bounds for p-norms

Gap_∞ (x,y) Problem

$x \in \{0, \ldots, B\}^n$ $y \in \{0, \ldots, B\}^n$

- Promise: $|x - y|_\infty \le 1$ or $|x - y|_\infty \ge B$
- Hard distribution non-product
- $\Omega(n/B^2)$ lower bound [Saks, Sun] [Bar-Yossef, Jayram, Kumar, Sivakumar]

Direct Sums

- $\mathrm{Gap}_\infty(x,y)$ doesn't have a hard product distribution, but has a hard distribution $\mu = \lambda^n$ in which the coordinate pairs $(x_1, y_1), \ldots, (x_n, y_n)$ are independent
  - w.pr. 1-1/n, (x_i, y_i) random subject to $|x_i - y_i| \leq 1$
  - w.pr. 1/n, (x_i, y_i) random subject to $|x_i - y_i| \geq B$
- Direct Sum: solving $\mathrm{Gap}_\infty(x,y)$ requires solving n single-coordinate sub-problems g
- In g, Alice and Bob have $J,K \in \{0, \ldots, B\}$, and want to decide if $|J-K| \leq 1$ or $|J-K| \geq B$

Direct Sum Theorem

- Let Π be the message from Alice to Bob, concatenated with Bob's output
- For $(X, Y) \sim \mu$, $I(\Pi ; X, Y) = H(X,Y) - H(X,Y \mid \Pi)$ is the information cost of the protocol
- [BJKS]: ?!?!?!?! why not measure $I(\Pi ; X, Y)$ when (X,Y) satisfy $|X - Y|_\infty \leq 1$?
  - Is $I(\Pi ; X, Y)$ large?
  - Let us go back to protocols correct on each X, Y w.h.p.
- Define $\mu = \lambda^n$, where $(X_i, Y_i) \sim \lambda$ is random subject to $|X_i-Y_i| \leq 1$
- $IC(g) = \inf_\psi I(\psi ; J, K)$, where ψ ranges over all 2/3-correct 1-way protocols for g, and $J,K \sim \lambda$

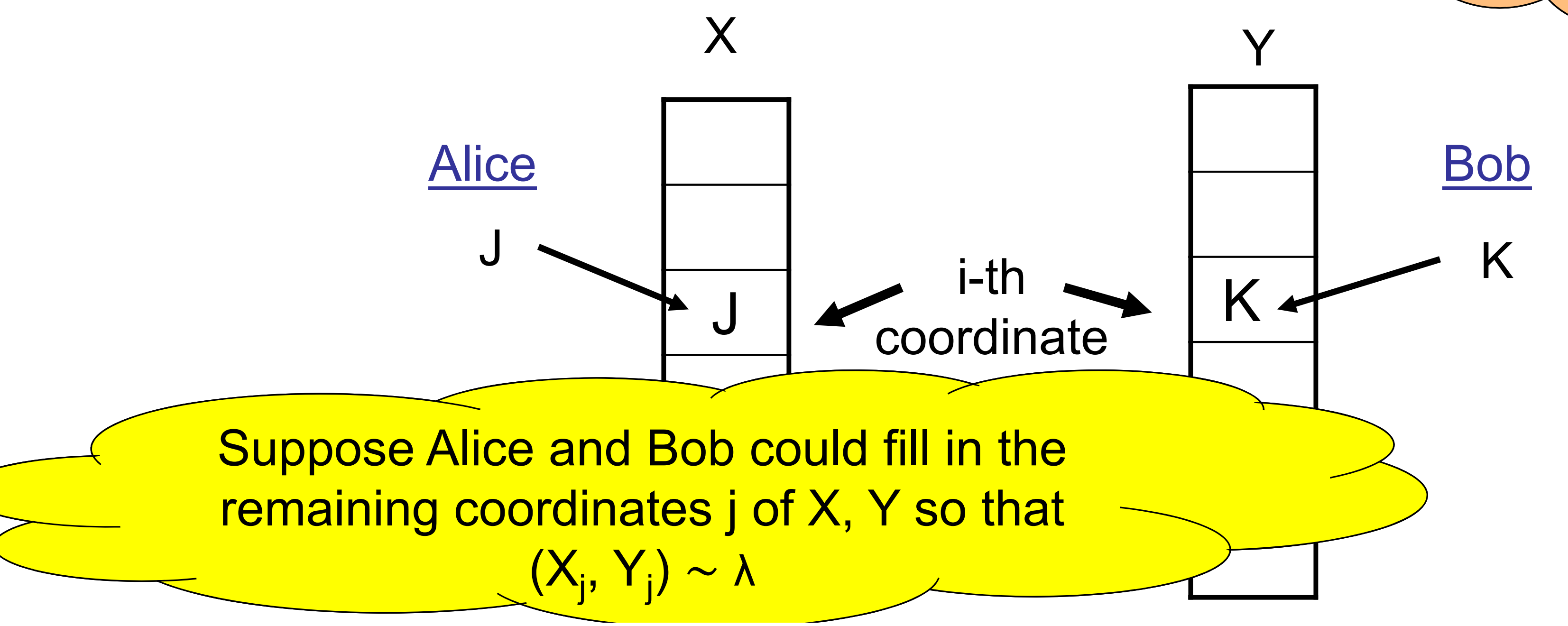
Is $I(\Pi ; X, Y) = \Omega(n) \cdot IC(g)$?

The Embedding Step

- $I(\Pi ; X, Y) \geq \sum_i I(\Pi ; X_i, Y_i)$

- We need to show $I(\Pi ; X_i, Y_i) \geq IC(g)$ for each i

Then we get a correct protocol for g!



Conditional Information Cost

- $(X_j, Y_j) \sim \lambda$ is not a product distribution

- [BJKS] Define $D = ((P_1, V_1) \ldots, (P_n, V_n))$:
  - P_j uniform on {Alice, Bob}
  - V_j uniform on $\{1, \ldots, B\}$ if P_j = Alice
  - V_j uniform on $\{0, \ldots, B-1\}$ if P_j = Bob
  - If P_j = Alice, then $Y_j = V_j$ and X_j is uniform on $\{V_j-1, V_j\}$
  - If P_j = Bob, then $X_j = V_j$ and Y_j is uniform on $\{V_j, V_j+1\}$

X and Y are independent conditioned on D!

- $I(\Pi ; X, Y \mid D) = \Omega(n) \cdot IC(g \mid (P,V))$ holds now!

- $IC(g \mid (P,V)) = \inf_\psi I(\psi ; J, K \mid (P,V))$, where ψ ranges over all 2/3-correct protocols for g, and $J,K \sim \lambda$

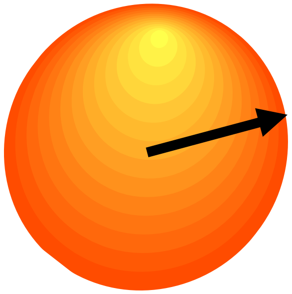
Primitive Problem

- Need to lower bound IC(g | (P,V))
- For fixed P = Alice and V = v, this is I(ψ ; K) where K is uniform over v-1, v
- From previous lecture: $I(\psi ; K) \geq D_{JS}(\psi_{v-1,v}, \psi_{v,v})$
- $IC(g \mid (P,V)) \geq E_{v \in \{1,...,B\}}[D_{JS}(\psi_{v-1,v}, \psi_{v,v})]/2$
  $+ E_{v \in \{0,...,B-1\}}[D_{JS}(\psi_{v,v}, \psi_{v,v+1})]/2$

Forget about distributions, let's move to unit vectors!

Hellinger Distance

- For distribution μ, let $\sqrt{\mu}$ be the vector with coordinate i equal to $\mu_i^{1/2}$



- $D_{JS}(\psi_{v-1,v}, \psi_{v,v}) \geq h(\psi_{v-1,v}, \psi_{v,v})^2$

$$(*) \quad IC(g \mid (P,V)) \geq E_{v \in \{1,...,B\}}[h^2(\psi_{v-1,v}, \psi_{v,v})]/2 + E_{v \in \{0,\ldots,B-1\}}[h^2(\psi_{v,v}, \psi_{v,v+1})]/2$$

- Properties
  - (Correctness) $h(\psi_{0,0}, \psi_{0,B})^2 = \Omega(1)$
  - (1-way Protocol) $\psi_{a,b}(m, out) = p_a(m) \cdot q_{b,m}(out)$
  - *(Pythagorean)* $h^2(\psi_{a,b}, \psi_{c,d}) \geq \frac{1}{2}\left(h^2(\psi_{a,b}, \psi_{a,d}) + h^2(\psi_{c,b}, \psi_{c,d})\right)$

Pythagorean Property

$$\frac{1}{2}\left(1 - h^2(\psi_{a,b}, \psi_{a,d}) + 1 - h^2(\psi_{c,b}, \psi_{c,d})\right)$$

$$= \frac{1}{2}\sum_{m_{\text{out}}} \left(\sqrt{p_a}(m) \cdot \sqrt{q_{b,m}(out)}\,\sqrt{p_a}(m)\,\sqrt{q_{d,m}(out)} + \sqrt{p_c}(m)\,\sqrt{q_{b,m}(out)}\,\sqrt{p_c}(m)\,\sqrt{q_{d,m}(out)} \right)$$

$$= \sum_{m_{\text{out}}} \frac{p_a(m)+p_c(m)}{2} \left(\sqrt{q_{b,m}(out)}\,\sqrt{q_{d,m}(out)}\right)$$

$$\geq \sum_{m_{\text{out}}} \sqrt{p_a}(m)\,\sqrt{p_c}(m)\sqrt{q_{b,m}(out)}\,\sqrt{q_{d,m}(out)}$$

$$= 1 - h^2(\psi_{a,b}, \psi_{c,d})$$

Lower Bounding the Primitive Problem

$\mathrm{IC}(g \mid (P, V)) \geq \mathrm{E}_{v \in \{1,...,B\}}[h^2(\psi_{v-1,v}, \psi_{v,v})]/2 + \mathrm{E}_{v\in\{0,\dots,B-1\}}[h^2(\psi_{v,v}, \psi_{v,v+1})]/2$

$= \frac{1}{2B}\left[\sum_{v\in\{1,\dots,B\}}\left|\sqrt{\psi_{v-1,v}} - \sqrt{\psi_{v,v}}\right|^2 + \sum_{v\in\{0,\dots,B-1\}}\left|\sqrt{\psi_{v,v}} - \sqrt{\psi_{v,v+1}}\right|^2\right]$

$\geq \frac{1}{4B^2}\left(\sum_{v\in\{1,\dots,B\}}\left|\sqrt{\psi_{v-1,v}} - \sqrt{\psi_{v,v}}\right| + \sum_{v\in\{0,1,\dots,B-1\}}\left|\sqrt{\psi_{v,v}} - \sqrt{\psi_{v,v+1}}\right|\right)^2$

$\geq \frac{1}{4B^2}\left(\sum_{v\in\{0,1,\dots,B-1\}}\left|\sqrt{\psi_{v,v}} - \sqrt{\psi_{v+1,v+1}}\right|\right)^2$

$\geq \frac{1}{4B^2}\left|\sqrt{\psi_{0,0}} - \sqrt{\psi_{B,B}}\right|^2$

$\geq \frac{1}{8B^2}\left(\left|\sqrt{\psi_{0,0}} - \sqrt{\psi_{0,B}}\right|^2 + \left|\sqrt{\psi_{B,0}} - \sqrt{\psi_{B,B}}\right|^2\right)$

$= \Omega(\frac{1}{B^2})$

Direct Sum Wrapup

- $\Omega(n/B^2)$ bound for $\mathrm{Gap}_\infty(x,y)$
- Similar argument gives $\Omega(n)$ bound for set disjointness [BJKS]
- Direct sums are nice, but often a problem can't be split into simpler smaller problems, e.g., no known embedding step in gap-Hamming

Nonnegative Matrix Factorization

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$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 5 & 2 & 2 \end{bmatrix}$$

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$$= UV$$

Main Question, Nonnegative Matrix Factorization

Question

Given matrix $A \in \mathbb{R}^{n \times n}$ and $k \geqslant 1$, is there an algorithm that can determine if there exist two matrices $U, V^\top \in \mathbb{R}^{n \times k}$,

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Or, are there any hardness results?

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- Equivalent to computing the nonnegative rank of A, $\mathrm{rank}_+(A)$
- Fundamental question in machine learning
- Applications
  - Text mining, Spectral data analysis, Scalable Internet distance prediction, Non-stationary speech denoising, Bioinformatics, Nuclear imaging, etc.

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$$\mathrm{rank}(A) = 3,$$

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$$\mathrm{rank}(A) = 3, \quad \text{but } \mathrm{rank}_+(A) = 4$$

Polynomial System Verifier

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[Renegar'92, Basu-Pollack-Roy'96]

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H : the bitsizes of the coefficients of the polynomials

In $(md)^{O(v)} \operatorname{poly}(H)$ time, can decide if there exists a solution to polynomial system P

Main Idea

1. Write $\min_{U,V^\top \in \mathbb{R}^{n\times k}, U,V \geqslant 0} \|UV - A\|_F^2$ as a polynomial system

   that has $\mathrm{poly}(k)$ variables and $\mathrm{poly}(n)$ constraints and degree

2. Use polynomial system verifier to solve it

Algorithm

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Output :

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in $n^{2^{O(k)}}$ time Arora-Ge-Kannan-Moitra'12

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in $2^{O(k^3)} n^{O(k^2)}$ time Moitra'13

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Determine: if there is a set of k numbers that sum to exactly $k/2$

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then k-SUM cannot be solved in $n^{o(k)}$ time

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Assume : Exponential Time Hypothesis
[Impagliazzo-Paturi-Zane'98]

Requires : $n^{\Omega(k)}$ time

Open Problems

- The upper bound is $n^{O(k^2)}$ while the lower bound is $n^{\Omega(k)}$ - what is the right answer?

Weighted Low Rank Approximation

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Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Weighted Low Rank Approximation

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : rank-k $\widehat{A} \in \mathbb{R}^{n \times n}$ s.t.

$$\left\| W \circ (\widehat{A} - A) \right\|_F^2$$

Weighted Low Rank Approximation

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : rank-k $\widehat{A} \in \mathbb{R}^{n \times n}$ s.t.

$$\| W \circ (\widehat{A} - A) \|_F^2 \leqslant (1 + \epsilon) \min_{\text{rank}-k \ A'} \| W \circ (A' - A) \|_F^2$$

$$\left\| \boxed{W} \circ \left(\boxed{\widehat{A}} - \boxed{A} \right) \right\|_F^2$$

Weighted Low Rank Approximation

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : rank-k $\widehat{A} \in \mathbb{R}^{n \times n}$ s.t.

$$\|W \circ (\widehat{A} - A)\|_F^2 \leqslant (1+\epsilon) \min_{\text{rank}-k \ A'} \underbrace{\|W \circ (A' - A)\|_F^2}_{\sum_{i,j} W_{i,j}^2 (A'_{i,j} - A_{i,j})^2}$$

$$\left\| W \circ (\widehat{A} - A) \right\|_F^2$$

Weighted Low Rank Approximation

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : rank-k $\widehat{A} \in \mathbb{R}^{n \times n}$ s.t.

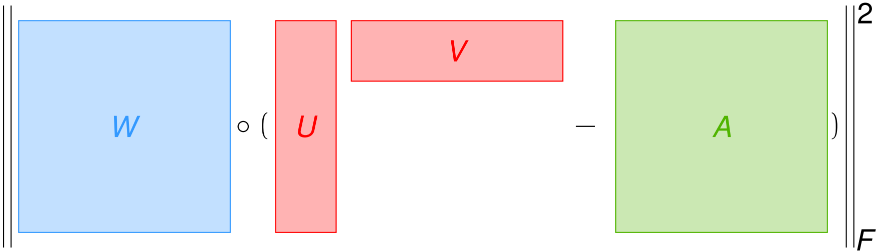
$$\|W \circ (\widehat{A} - A)\|_F^2 \leqslant (1+\epsilon) \overbrace{\min_{\text{rank}-k\ A'} \underbrace{\|W \circ (A' - A)\|_F^2}_{\sum_{i,j} W_{i,j}^2 (A'_{i,j} - A_{i,j})^2}}^{\text{OPT}}$$

$$\left\| \boxed{W} \circ \left(\boxed{\widehat{A}} - \boxed{A} \right) \right\|_F^2$$

Weighted Low Rank Approximation

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t.
$\| W \circ (UV - A) \|_F^2 \leqslant (1 + \epsilon)\,\mathrm{OPT}$



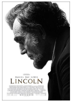
Motivation

Motivation



DIVIDED WE FALL
CIVIL WAR
MAY 6

JIM CARREY
MASK
FROM ZERO TO HERO

HANGOVER

LINCOLN


ZOOTOPIA



LORD OF THE RINGS

Motivation

Motivation

Action Comedy Historical Cartoon Magical

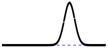
Motivation

Action Comedy Historical Cartoon Magical

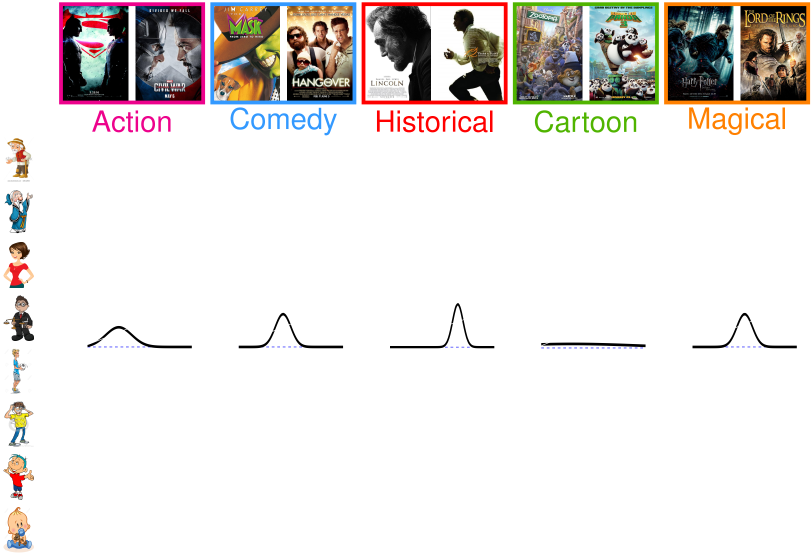
9 8

Motivation

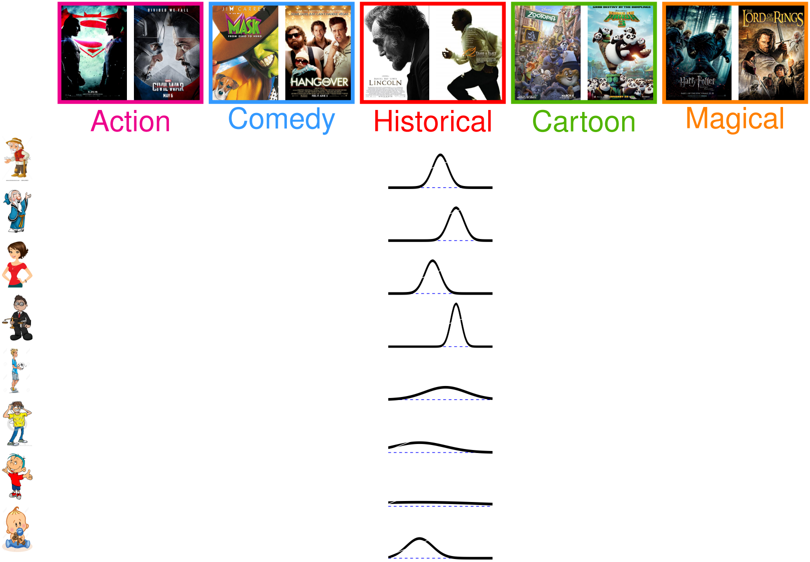
Action Comedy Historical Cartoon Magical



Motivation



Motivation



Motivation

Action | Comedy | Historical | Cartoon | Magical

	Action	
	$\frac{1}{5}$	$\frac{1}{5}$
	$\frac{1}{1}$	$\frac{1}{1}$
	$\frac{1}{1}$	$\frac{1}{1}$
	$\frac{1}{5}$	$\frac{1}{5}$
	$\frac{1}{1}$	$\frac{1}{1}$
	$\frac{1}{4}$	$\frac{1}{4}$
	$\frac{1}{6}$	$\frac{1}{6}$
	$\frac{1}{1}$	$\frac{1}{1}$

Motivation

Action		Comedy		Historical	Cartoon	Magical
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$			
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$			
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$			
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$			
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$			
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$			

Motivation

Action		Comedy		Historical		Cartoon		Magical	
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$				
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{6}$	$\frac{1}{6}$				
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$				
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$				
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{7}$				
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$				
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$				
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{1}$	$\frac{1}{1}$				

Motivation

	Action		Comedy		Historical		Cartoon		Magical
	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	
	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{1}$	$\frac{1}{1}$	
	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	
	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{1}$	$\frac{1}{1}$	
	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{5}$	
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$	
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	
	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	

Motivation

Action		Comedy		Historical		Cartoon		Magical	
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$

Motivation

Action Comedy Historical Cartoon Magical

Motivation

Action Comedy Historical Cartoon Magical

8 8

8 8

Motivation

Action		Comedy		Historical		Cartoon		Magical	
				8	8				
				8	8				
				4	4				
				4	4				

Motivation

Action Comedy Historical Cartoon Magical

8	8
8	8
4	4
4	4
1	1
1	1

Motivation

Action		Comedy		Historical		Cartoon		Magical	
				8	8				
				8	8				
				4	4				
				4	4				
				1	1				
				1	1				
				0	0				
				0	0				

Motivation

Action		Comedy		Historical		Cartoon		Magical	
				8	8				
				8	8				
				4	4				
				4	4				
				1	1				
				1	1				
				0	0	8	8		
				0	0	8	8		

Motivation

Action		Comedy		Historical		Cartoon		Magical	
				8	8				
				8	8				
				4	4				
				4	4				
				1	1	4	4		
				1	1	4	4		
				0	0	8	8		
				0	0	8	8		

Motivation

Action Comedy Historical Cartoon Magical

8	8		
8	8		
4	4	2	2
4	4	2	2
1	1	4	4
1	1	4	4
0	0	8	8
0	0	8	8

Motivation

Action Comedy Historical Cartoon Magical

8	8	1	1
8	8	1	1
4	4	2	2
4	4	2	2
1	1	4	4
1	1	4	4
0	0	8	8
0	0	8	8

Matrix Completion

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n\times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n\times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n \times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Output : $A_{\bar{\Omega}}$ s.t. $\mathrm{rank}(A) = k$

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n\times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Output : $A_{\bar{\Omega}}$ s.t. $\mathrm{rank}(A) = k$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n\times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Output : $A_{\bar{\Omega}}$ s.t. $\mathrm{rank}(A) = k$

Algorithms : Candes-Recht'09, Candes-Recht'10, Keshavan'12
Hardt-Wootters'14, Jain-Netrapalli-Sanghavi'13
Hardt'15, Sun-Luo'15

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n \times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Output : $A_{\bar{\Omega}}$ s.t. $\mathrm{rank}(A) = k$

Algorithms : Candes-Recht'09, Candes-Recht'10, Keshavan'12
Hardt-Wootters'14, Jain-Netrapalli-Sanghavi'13
Hardt'15, Sun-Luo'15

Hardness : Peeters'96, Gillis-Glineur'11
Hardt-Meka-Raghavendra-Weitz'14

More Real-life Datasets

More Real-life Datasets

RateBeer

Tasters

Beers

Heaven Or Helles
Boston Beer Company (Samuel Adams)
Munich Helles Lager / 5.00% ABV

3.71/5 rDev +0.5% | Avg: 3.69
look: 3.25 | smell: 3.5 | taste: 4 | feel: 3.75 | overall: 3.5

Pours a slightly hazy golden straw color with a pillowy 1-1 1/2 inch head that slowly dissipates to slightly less than 1 finger of head with good lacing. The nose had aromas of bread malt and a spiciness I belie cv is pepper. The mouthfeel was lightly carbonated with a richness/sweetness. A slight bitterness on the back end rounded out the flavor palate and struck a nice balance. I found it very satisfying and I'd go as far as saying it would make a nice session beer. The only thing I don't like is that it is a limited release.

★ 533 characters

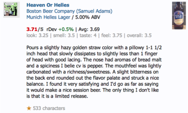
Ratings

More Real-life Datasets

RateBeer

Tasters

Beers

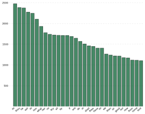


Ratings

Documents

Words

Books



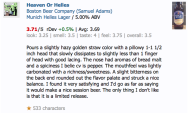
Frequency

More Real-life Datasets

RateBeer

Tasters

Beers

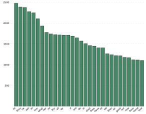


Ratings

Documents

Words

Books



Frequency

Amazon

Users

Products

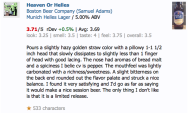
Ratings

More Real-life Datasets

RateBeer

Tasters

Beers

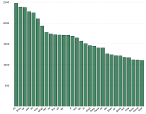


Ratings

Documents

Words

Books



Frequency

Amazon

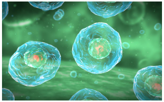
Users

Products

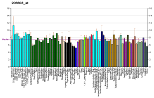
Ratings

Biology

Genes



Cells



Levels

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem
  - W has r distinct rows and columns

$$W = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 4 \\ 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 7 & 7 & 7 & 4 & 4 & 6 \end{bmatrix}, r = 3$$

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem
  - W has r distinct rows and columns
  - W has r distinct columns

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 3 & 1 & 1 & 5 \\ 4 & 4 & 4 & 1 & 1 & 1 \\ 5 & 5 & 5 & 1 & 1 & 3 \\ 6 & 6 & 6 & 1 & 1 & 5 \end{bmatrix}, r = 3$$

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem
  - W has r distinct rows and columns
  - W has r distinct columns
  - W has rank at most r

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}, r = 3$$

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem
  - W has r distinct rows and columns
  - W has r distinct columns
  - W has rank at most r
- Hardness for Weighted low rank approximation(WLRA) problem

r Distinct Rows and Columns

r Distinct Rows and Columns

Given : $\quad A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

r Distinct Rows and Columns

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 4 \\ 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 7 & 7 & 7 & 4 & 4 & 6 \end{bmatrix}, r = 3$$

r Distinct Rows and Columns

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Output :

r Distinct Rows and Columns

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 4 \\ 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 7 & 7 & 7 & 4 & 4 & 6 \end{bmatrix}, r = 3$$

Output : rank-k $\widehat{A} \in \mathbb{R}^{n \times n}$ s.t.

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Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 4 \\ 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 7 & 7 & 7 & 4 & 4 & 6 \end{bmatrix}, r = 3$$

Output : rank-k $\widehat{A} \in \mathbb{R}^{n \times n}$ s.t.

$$\|W \circ (A - \widehat{A})\|_F^2 \leqslant (1 + \epsilon) \min_{\text{rank}-k\ A'} \|W \circ (A - A')\|_F^2$$

r Distinct Rows and Columns

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 4 \\ 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 7 & 7 & 7 & 4 & 4 & 6 \end{bmatrix}, r = 3$$

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r Distinct Rows and Columns

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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$$\|W \circ (A - \widehat{A})\|_F^2 \leqslant (1+\epsilon)\mathrm{OPT}$$

r Distinct Rows and Columns

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 4 \\ 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 7 & 7 & 7 & 4 & 4 & 6 \end{bmatrix}, r = 3$$

Output : rank-k $\widehat{A} \in \mathbb{R}^{n\times n}$ s.t.
$\|W \circ (A - \widehat{A})\|_F^2 \leqslant (1+\epsilon)\mathrm{OPT}$
with prob. $9/10$

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Random-4SAT Hypothesis

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[Feige'02, Goerdt-Lanka'04]

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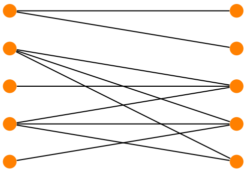

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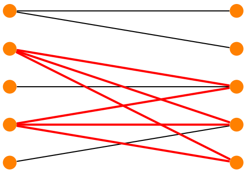

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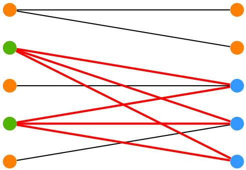

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$$A_{i,j} = \begin{cases} 1 \\ 0 \end{cases} \quad W_{i,j} = \begin{cases} 1 & \text{if edge } (U_i, V_j) \text{ exists} \\ n^6 & \text{otherwise} \end{cases}$$

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Requires : $2^{\Omega(r)}$ time

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It takes $(md)^{O(v)}\,\mathrm{poly}(H)$ time to decide if there exists a solution to polynomial system P

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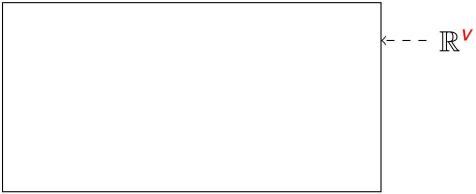

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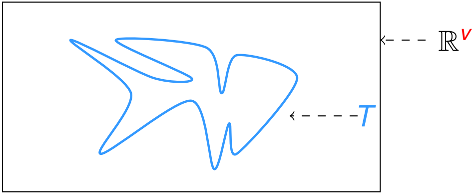

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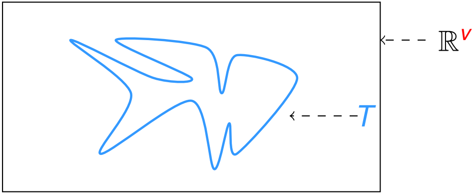

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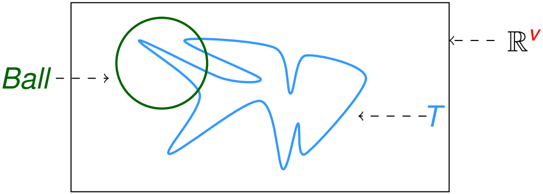

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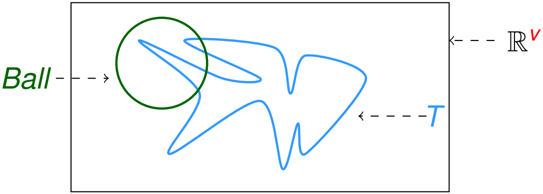

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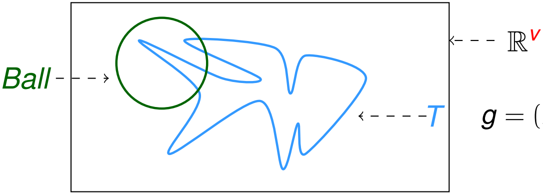

$g = (x_1 x_2 - 1)^2 + x_2^2$

Multiple Regression Sketch

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Warmup, inefficient WLRA Algorithm

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Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Time : $2^{\Omega(nk)}$

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2. write polynomial $g(x_1, \cdots, x_{2nk}) = \|W \circ (A - UV)\|_F^2$
3. pick $C \in [L^-, L^+]$, run polynomial verifier $g(x) \leqslant C$
4. optimize C by binary search over $[L^-, L^+]$

Time : $2^{\Omega(nk)}$

How can we do better?

Warmup, inefficient WLRA Algorithm

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : rank-k $\widehat{A}$ s.t. $\| W \circ (A - \widehat{A}) \|_F^2 \leqslant (1 + \epsilon)\,\mathrm{OPT}$

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polynomial verifier runs in $(\#\,\text{constraints} \cdot \text{degree})^{O(\#\,\text{variables})}$

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lower bound on cost $(\#\,\text{constraints})^{-\text{degree}^{O(\#\,\text{variables})}}$

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write a polynomial with few #variables, i.e. $\mathrm{poly}(kr/\epsilon)$

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polynomial verifier runs in $(\#\,\text{constraints} \cdot \text{degree})^{O(\#\,\text{variables})}$

lower bound on cost $(\#\,\text{constraints})^{-\text{degree}^{O(\#\,\text{variables})}}$

write a polynomial with few #variables, i.e. $\mathrm{poly}(kr/\epsilon)$

without blowing up degree and #constraints too much

Main Idea

To reduce the number of variables to $\mathrm{poly}(kr/\epsilon)$:

1. Multiple regression sketch with $O(k/\epsilon)$ rows
2. Weight matrix W has rank at most r

Guess a Sketch

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$
$A_{ij} \in \{0, \pm 1, \pm 2, \cdots, \pm \Delta\}$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W_{ij} \in \{0, 1, 2, \cdots, \Delta\}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : U, $V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

D_{W_j} be diagonal matrix with vector W_j

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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D_{W_j} be diagonal matrix with vector W_j

$$W = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{matrix}} & \begin{matrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{matrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 2 \\ 4 \\ 4 \\ 6 \\ 6 \end{matrix} & \begin{matrix} 2 \\ 3 \\ 4 \\ 4 \\ 5 \\ 6 \end{matrix} \end{bmatrix} \quad D_{W_1} = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\| W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

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$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad D_{W_2} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \quad D_{W_3} = \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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D_{W_j} be diagonal matrix with vector W_j

$$W = \begin{bmatrix} 1 & 1 & 1 & \boxed{2} & 2 & 2 \\ 0 & 1 & 2 & \boxed{1} & 2 & 3 \\ 1 & 1 & 3 & \boxed{2} & 4 & 4 \\ 0 & 0 & 4 & \boxed{0} & 4 & 4 \\ 1 & 0 & 5 & \boxed{1} & 6 & 5 \\ 0 & 0 & 6 & \boxed{0} & 6 & 6 \end{bmatrix} \quad D_{W_4} = \begin{bmatrix} 2 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Algorithm : W_j be jth column of W
D_{W_j} be diagonal matrix with vector W_j

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad D_{W_5} = \begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 6 & \\ & & & & & 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad D_{W_6} = \begin{bmatrix} 2 & & & & & \\ & 3 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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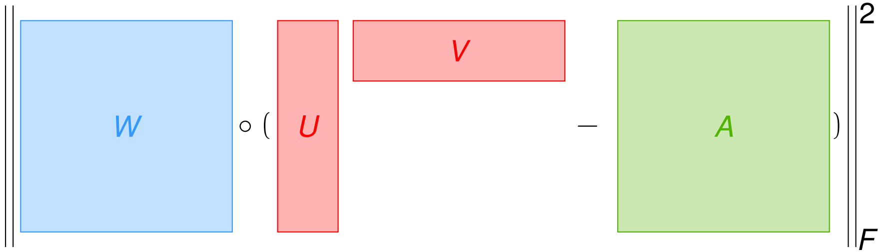
Algorithm :



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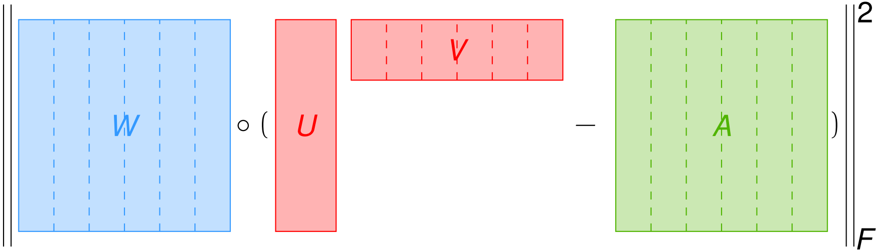

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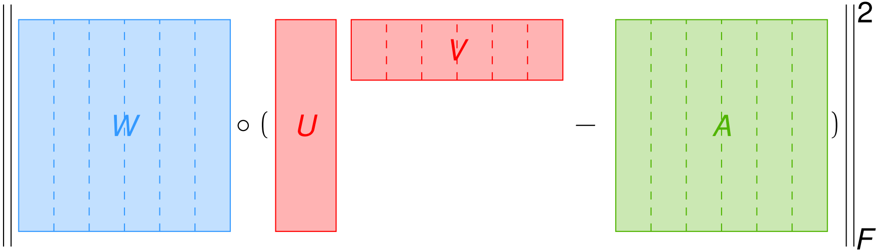
Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$



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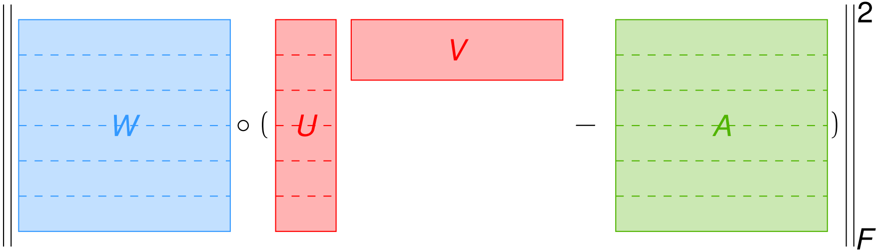

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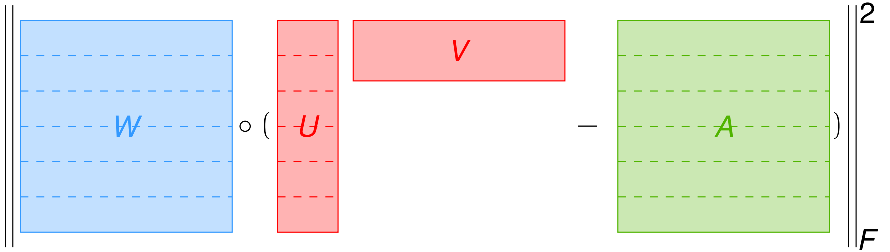

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Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

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$\downarrow$ Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$$

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Guess a Sketch

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$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$ $\qquad$ $\sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$ $\quad$ $\sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_1} U = \boxed{\quad S \quad} \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_2} U = \boxed{\quad S \quad} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\| W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2 \longleftrightarrow \sum_{i=1}^{n} \| U^i V D_{W^i} - A^i D_{W^i} \|_2^2$

$\downarrow$ Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$ $\downarrow$

$$\sum_{j=1}^{n} \| S D_{W_j} U V_j - S D_{W_j} A_j \|_2^2 \qquad \sum_{i=1}^{n} \| U^i V D_{W^i} T - A^i D_{W^i} T \|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_3} U = \boxed{\quad S \quad} \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_4} U = S \begin{bmatrix} 2 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} U$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

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Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$ $\quad$ $\sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_5} U = \boxed{S} \begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 6 & \\ & & & & & 6 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\| W \circ (UV - A)\|_F^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2 \longleftrightarrow \sum_{i=1}^{n} \| U^i V D_{W^i} - A^i D_{W^i} \|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \| S D_{W_j} U V_j - S D_{W_j} A_j \|_2^2 \qquad \sum_{i=1}^{n} \| U^i V D_{W^i} T - A^i D_{W^i} T \|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_6} U = \boxed{\quad S \quad} \begin{bmatrix} 2 & & & & & \\ & 3 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $SD_{W_j}U \in \mathbb{R}^{t\times k}$ and $VD_{W^i}T \in \mathbb{R}^{k\times t}$

Create $t \times k$ variables for each of n $SD_{W_j}U$ s?

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

$\downarrow$ Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$ $\downarrow$

$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$ $\quad$ $\sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

Create $t \times k$ variables for each of n $S D_{W_j} U$ s?

$$n \times t \times k$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$ $\quad$ $\sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

Create $t \times k$ variables for each of n $S D_{W_j} U$ s?

$? \times t \times k$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

$\downarrow$ Sketch by Gaussian S, $T^\top \in \mathbb{R}^{t \times n}$ $\downarrow$

$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$ $\qquad$ $\sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$

Guess $S D_{W_j} U \in \mathbb{R}^{t \times k}$ and $V D_{W^i} T \in \mathbb{R}^{k \times t}$

Create $t \times k$ variables for each of n $S D_{W_j} U$ s?

$r \times t \times k$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : U, $V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad \overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 0 & 4 \\ 1 & 0 & 5 \\ 0 & 0 & 6 \end{bmatrix}}^{\text{column span of } W}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : U, $V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\1\\0\\1\\0\end{matrix}} & \begin{matrix}1\\1\\1\\0\\0\\0\end{matrix} & \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} & \begin{matrix}2\\1\\2\\0\\1\\0\end{matrix} & \begin{matrix}2\\2\\4\\4\\6\\6\end{matrix} & \begin{matrix}2\\3\\4\\4\\5\\6\end{matrix} \end{bmatrix}$$

$$W_1 = W_1$$

$$\begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W = \begin{bmatrix} 1 & \boxed{1} & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}$$

$$W_2 = W_2$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\| W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_3 = W_3$$

$$W = \begin{bmatrix} 1 & 1 & \boxed{1} & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_4 = W_1 + W_2$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\| W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_5 = W_1 + W_3$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & \boxed{2} & 2 \\ 0 & 1 & 2 & 1 & \boxed{2} & 3 \\ 1 & 1 & 3 & 2 & \boxed{4} & 4 \\ 0 & 0 & 4 & 0 & \boxed{4} & 4 \\ 1 & 0 & 5 & 1 & \boxed{6} & 5 \\ 0 & 0 & 6 & 0 & \boxed{6} & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\| W \circ (UV - A) \|_F^2 \leqslant (1 + \epsilon)\, \mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_6 = W_2 + W_3$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \\ 4 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}_{\sum_{j=1}^{n} \|D_{W_j} UV_j - D_{W_j} A_j\|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}_{\sum_{j=1}^{n} \|D_{W_j} UV_j - D_{W_j} A_j\|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \|D_{W_j} UV_j - D_{W_j} A_j\|_2^2 \quad \xrightarrow{1+\epsilon} \quad \text{sketch} \sum_{j=1}^{n} \|SD_{W_j} UV_j - SD_{W_j} A_j\|_2^2$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}_{\sum_{j=1}^n \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$$

$$\text{sketch } \sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \quad 1+\epsilon$$

create variables for $S D_{W_j} U, \forall j \in [r]$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}_{\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$$

$1+\epsilon$

sketch $\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

create $t \times k$ variables for $S D_{W_1} U$

$$S \qquad \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} \qquad U$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\underbrace{\| W \circ (UV - A) \|_F^2}_{\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2$$

$1 + \epsilon$

sketch $\sum_{j=1}^{n} \| S D_{W_j} U V_j - S D_{W_j} A_j \|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

create $t \times k$ variables for $S D_{W_2} U$

S

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

U

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\| W \circ (UV - A) \|_F^2}_{\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

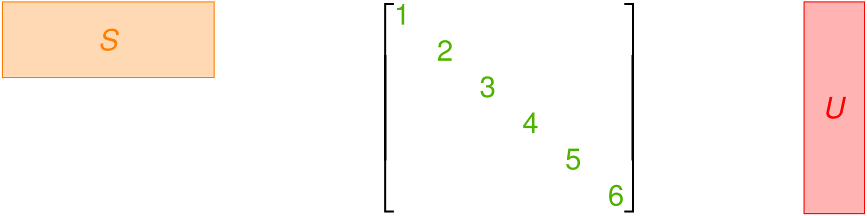
$$\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2$$

$1+\epsilon$

sketch $\sum_{j=1}^{n} \| S D_{W_j} U V_j - S D_{W_j} A_j \|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

create $t \times k$ variables for $S D_{W_3} U$



Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$$

$1+\epsilon$

sketch $\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

write $S D_{W_4} U$ as $S D_{W_1} U + S D_{W_2} U$

S

$$\begin{bmatrix} 2 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix}$$

U

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}_{\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$$

$1+\epsilon$

sketch $\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

write $S D_{W_4} U$ as $S D_{W_1} U + S D_{W_2} U$

$$S \begin{bmatrix} 1+1 & & & & & \\ & 0+1 & & & & \\ & & 1+1 & & & \\ & & & 0+0 & & \\ & & & & 1+0 & \\ & & & & & 0+0 \end{bmatrix} U$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\| W \circ (UV - A) \|_F^2}_{\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2$$

$1+\epsilon$

sketch $\sum_{j=1}^{n} \| S D_{W_j} U V_j - S D_{W_j} A_j \|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

write $S D_{W_4} U$ as $S D_{W_1} U + S D_{W_2} U$

$$S \left(\begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} \right) U$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$

$1+\epsilon$

sketch $\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

write $S D_{W_5} U$ as $S D_{W_1} U + S D_{W_3} U$

S

$$\begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 6 & \\ & & & & & 6 \end{bmatrix}$$

U

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}, W \in \mathbb{R}^{n \times n}, r \in \mathbb{N}, k \in \mathbb{N}, \epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\underbrace{\| W \circ (UV - A) \|_F^2}_{\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2 \qquad 1+\epsilon$$

sketch $\sum_{j=1}^{n} \| S D_{W_j} U V_j - S D_{W_j} A_j \|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

write $S D_{W_5} U$ as $S D_{W_1} U + S D_{W_3} U$

$$S \begin{bmatrix} 1+1 & & & & & \\ & 0+2 & & & & \\ & & 1+3 & & & \\ & & & 0+4 & & \\ & & & & 1+5 & \\ & & & & & 0+6 \end{bmatrix} U$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\underbrace{\|W \circ (UV - A)\|_F^2} = \sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$$

$1+\epsilon$

sketch $\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

write $S D_{W_5} U$ as $S D_{W_1} U + S D_{W_3} U$

$$S \left(\begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix} \right) U$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\underbrace{\| W \circ (UV - A) \|_F^2}_{}\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

$$\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2$$

$1 + \epsilon$

sketch $\sum_{j=1}^{n} \| S D_{W_j} U V_j - S D_{W_j} A_j \|_2^2$

create variables for $S D_{W_j} U, \forall j \in [r]$

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S

$$\begin{bmatrix} 2 & & & & & \\ & 3 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix}$$

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$$\boxed{S} \quad \begin{bmatrix} 1+1 & & & & & \\ & 1+2 & & & & \\ & & 1+3 & & & \\ & & & 0+4 & & \\ & & & & 0+5 & \\ & & & & & 0+6 \end{bmatrix} \quad \boxed{U}$$

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$$(\#\,\text{constraints} \cdot \text{degree})^{O(\#\,\text{variables})} = (O(n) \cdot O(nk))^{O(krt)}$$

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