1. What is an algorithm with the smallest number of comparisons you can find for outputting both the maximum and the minimum of \( n \) numbers?

2. Suppose \( H \) is a 3-universal family of hash functions \( h \), where each function \( h \in H \) is a mapping from \( \{1, 2, \ldots, M\} \) to \( \{1, 2, \ldots, N\} \). Show that \( H \) is also a 2-universal family of hash functions.

3. Consider the following data stream algorithm for estimating counts of items. We choose a universal hash function \( h : \Sigma \rightarrow \{0, 1, 2, \ldots, r - 1\} \) and a 2-universal hash function \( s : \Sigma \rightarrow \{-1, 1\} \). We initialize an array \( A \) of length \( r \) to be all zeros. When we see the stream item \( a_i \), we set \( A[h(a_i)] = A[h(a_i)] + s(a_i) \). At the end of the stream, for an element \( e \in \Sigma \), suppose we output \( \text{est}(e) = s(e) \cdot A[h(e)] \).

   (a) Show that the expected value \( \mathbb{E}[\text{est}(e)] = \text{count}(e) \), where \( \text{count}(e) \) is the number of occurrences of \( e \) in the stream.

   (b) Recall the variance \( \text{Var}[X] \) of a random variable \( X \) is \( \mathbb{E}[(X - \mathbb{E}[X])^2] \), which is also equal to \( \mathbb{E}[X^2] - \mathbb{E}^2[X] \). The smaller the variance is, the more likely your random variable is to be close to its expected value. Show \( \text{Var}[\text{est}(e)] \leq \frac{1}{r} \sum_{e' \in \Sigma} \text{count}^2(e') \). It might help to use that for independent random variables \( X \) and \( Y \), it holds that \( \mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y] \).