Lecture 20: The Multiplicative Weights Algorithm

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The Experts Problem

- n “experts” try to predict an outcome on each day

- Expert = someone with an opinion, not necessarily someone who knows anything

- For example, the experts could try to predict the stock market

Expt 1	Expt 2	Expt 3	neighbor’s dog	truth
down	up	up	up	up
down	up	up	down	down
…	…	…	…	…

The Experts Problem

- n “experts” predict an outcome on each of T days, t = 1, …, T
- On day t, the i-th expert predicts outcome out_i^t
- On day t, you see $\mathrm{out}_1^t, \ldots, \mathrm{out}_n^t$ and make your prediction guess^t
- Then you see the actual outcome out^t on day t
- You are correct if $\mathrm{guess}^t = \mathrm{out}^t$ and wrong otherwise

The Experts Problem

- Goal: if the best expert is wrong on M days, you want to be wrong on at most M days, plus a little bit

- Don't make assumptions on the input

- Don't assume future looks like the past

- Application: experts predict stock market, you want to do as well as the best single expert in hindsight

How should you choose your guess on each day?

Simpler Question

- Suppose at least one expert is perfect, i.e., never makes a mistake
  - Don't know which one
- Suppose each expert predicts one of two values: 0 or 1
  - E.g., stock market will go up or down
- Can we find a strategy that makes no more than $\lceil \lg_2 n \rceil$ mistakes?
- Majority-and-halving: On each day, take the majority vote of all experts
  - Each time you're wrong, you can remove at least half the experts
  - After $\lceil \lg_2 n \rceil$ mistakes you're left with the perfect expert
- Same guarantee if experts predict more than 2 values
  - You choose most frequent prediction. If wrong, at least half the experts are wrong

Can You Do Better?

- Claim: in the worst case, any strategy makes at least $\lg_2 n$ mistakes
- Proof: adversary method
- Day 1: make the first n/2 experts say 0, and the second n/2 experts say 1
  - If predictor outputs 0, then say the best expert outputs 1
  - If predictor outputs 1, then say the best expert outputs 0
  - Perfect expert is either in [1, n/2] or in [n/2+1, n]
- Day 2: in each interval [1, n/2] and [n/2+1, n], make first half of the experts say 0 and second half of the experts say 1
  - If predictor outputs 0, then say the best expert outputs 1
  - If predictor outputs 1, then say the best expert outputs 0
  - Perfect expert is either in [1, n/4], [n/4+1, n/2], [n/2+1, 3n/4], or [3n/4+1, n]
- …
- Any strategy is incorrect on at least $\lg_2 n$ days

No Perfect Expert

- Suppose best expert makes M mistakes
- How can we guarantee we make at most (M+1)($\log_2 n + 1$) mistakes?
- Run Majority-and-Halving, but after throwing away all experts, bring them all back in and start over
- In each “phase”, each expert makes at least 1 mistake, and you make at most $\log_2 n + 1$ mistakes
- At most M finished phases, plus the last unfinished one

Doing Better

- If best expert makes M mistakes, we make at most (M+1)$(\log_2 n + 1) =$ $O(M \log_2 n)$ mistakes

- Can't do better than best expert, who makes M mistakes
  - Suppose only one expert who always says 1 and is wrong M times

- Can't do better than $\log_2 n$ mistakes

- But can we make at most $\approx M + \log_2 n$ mistakes instead of $\approx M \cdot \log_2 n$?

Weighted Majority Algorithm

- Throwing away an expert when it makes a mistake is too drastic
- Assign weight w_i to i-th expert. Initialize all weights to 1
- On t-th day, compute sum of weights of experts who say 0, and sum of weights of experts who say 1
- Choose outcome with larger weight
- If an expert is wrong on day t, cut its weight in half

Weighted Majority Algorithm

- Theorem: If the best expert makes M mistakes, then the weighted majority algorithm makes at most $2.41(\mathrm{M} + \log_2 \mathrm{n})$ mistakes!

- Proof: Let $\Phi = \sum_i w_i$. Initially $\Phi = n$

- When we make a mistake, $\Phi_{new} \leq \frac{3}{4} \cdot \Phi_{old}$
  - At least half of the weight (which made the majority prediction) gets halved (because it made a mistake)

- If we don't make a mistake, $\Phi_{new} \leq \Phi_{old}$

Weighted Majority Algorithm

- If we've made m mistakes so far, $\Phi_{\text{final}} \leq \left(\frac{3}{4}\right)^{m} \cdot \Phi_{\text{init}} = \left(\frac{3}{4}\right)^{m} \cdot n$
- But best expert i^* makes at most M mistakes, so $\Phi_{\text{final}} \geq w_{i^*} \geq \left(\frac{1}{2}\right)^{M}$
- So $\left(\frac{1}{2}\right)^{M} \leq \Phi_{\text{final}} \leq \left(\frac{3}{4}\right)^{m} \cdot n$, or $\left(\frac{4}{3}\right)^{m} \leq 2^{M} \cdot n$
- Taking logs, $m \leq \frac{M+\log_2 n}{\log_2\left(\frac{4}{3}\right)} = 2.41(M + \log_2 n)$
- If best expert makes a mistake 10% of the time, we make a mistake 24% of the time (plus $\log_2 n$ which is negligible with enough days)

Improved Weighted Majority Algorithm

- Only change: if an expert is wrong on day t, multiply its weight by $1-\epsilon$
- Still choose outcome given by the majority weight of experts in each day
- Theorem: If the best expert makes M mistakes, then the weighted majority algorithm makes at most $2(1+\epsilon)M + O(\frac{\log_2 n}{\epsilon})$ mistakes

Improved Weighted Majority Algorithm

- Each time we make a mistake, $\Phi_{\text{new}} \leq (1 - \frac{\epsilon}{2}) \cdot \Phi_{\text{old}}$
  - At least half of the weight gets scaled by $1 - \epsilon$

- If we've made m mistakes so far, $\Phi_{\text{final}} \leq \left(1 - \frac{\epsilon}{2}\right)^{m} \cdot \Phi_{\text{init}} = \left(1 - \frac{\epsilon}{2}\right)^{m} n$
- $\Phi_{\text{final}} \geq w_{i^*} \geq (1 - \epsilon)^{M}$

- $(1 - \epsilon)^{M} \leq \Phi_{\text{final}} \leq \left(1 - \frac{\epsilon}{2}\right)^{m} n$ or $\frac{1}{\left(1-\frac{\epsilon}{2}\right)^{m}} \leq \frac{n}{(1-\epsilon)^{M}}$

- So $m \ln \frac{1}{1-\frac{\epsilon}{2}} \leq M \cdot \ln \frac{1}{1-\epsilon} + \ln n$
  - Use $\ln \frac{1}{1-\frac{\epsilon}{2}} \geq \frac{\epsilon}{2}$ and $\ln \frac{1}{1-\epsilon} \leq \epsilon + \epsilon^2$ for $\epsilon \in [0, \frac{1}{2}]$, and divide both sides by $\frac{2}{\epsilon}$
  - $m \leq 2M(1 + \epsilon) + \frac{2 \ln n}{\epsilon}$

Lower Bound for Deterministic Algorithms

- Theorem: If the best expert makes M mistakes, then the weighted majority algorithm makes at most $2(1+\epsilon)M + O(\frac{\log_2 n}{\epsilon})$ mistakes!
- If best expert is wrong 10% of the time, we're wrong 20% of the time
- 2-approximation is best possible for deterministic algorithms:
  - Suppose we have two experts - one always says 0 and one always says 1
  - If algorithm is deterministic, the adversary knows what prediction it will make on each day, so it can choose the opposite outcome
  - So algorithm incorrect on all days, but one expert is correct on at least half of the days

Randomized Weighted Majority Algorithm

- Assign weight $\mathrm{w_i}$ to i-th expert. Initialize all weights to 1
- On each day, predict 1 with probability $\frac{\sum_{\mathrm{i\ says\ 1}} \mathrm{w_i}}{\sum_{\mathrm{i}} \mathrm{w_i}}$, and predict 0 otherwise
- Equivalently, pick a random expert i with probability $\frac{\mathrm{w_i}}{\sum_{\mathrm{j}} \mathrm{w_j}}$ and choose that expert's outcome
- When an expert makes a mistake, multiply its weight by $1 - \epsilon$

Randomized Weighted Majority Algorithm

- Theorem: If the best expert makes M mistakes, then the expected number of mistakes of the randomized weighted majority algorithm makes at most $(1+\epsilon)M + \frac{\ln n}{\epsilon}$
- Let $\Phi = \sum_i w_i$. Initially $\Phi = n$
- Having fixed the outcome on all days, the potential varies deterministically

Randomized Weighted Majority Algorithm

- Let F_t be the fraction of total weight on the t-th day on experts that make a mistake on that day
- The expected number of mistakes we make is $\sum_t F_t$
- On day t: $\Phi_{new} = \Phi_{old} \cdot (1 - F_t) + \Phi_{old} \cdot F_t(1 - \epsilon) = \Phi_{old}\,(1 - \epsilon \cdot F_t)$
- $\Phi_{final} \leq n \cdot \prod_t (1 - \epsilon \cdot F_t) \leq n \cdot e^{-\epsilon \sum_t F_t}$ using that $1 + x \leq e^x$ for all x
- Also, $\Phi_{final} \geq (1 - \epsilon)^M$

Randomized Weighted Majority Algorithm

- Have shown: $(1-\epsilon)^M \leq \Phi_{\text{final}} \leq n \cdot e^{-\epsilon \sum_t F_t}$
- Taking natural logs, $\epsilon \sum_t F_t \leq M \ln \frac{1}{1-\epsilon} + \ln n$
- Using $\ln \frac{1}{1-\epsilon} \leq \epsilon + \epsilon^2$ for $\epsilon \in [0, \frac{1}{2}]$, and dividing both sides by ϵ we get:

Expected number of mistakes = $\sum_t F_t \leq M(1+\epsilon) + \frac{\ln n}{\epsilon}$

Understanding the Error Rate

- Expected number of mistakes = $\sum_t F_t \leq M(1+\epsilon) + \frac{\ln n}{\epsilon}$
- Best expert makes at most T mistakes, so $\sum_t F_t \leq M + \epsilon T + \frac{\ln n}{\epsilon}$
- Let M/T be optimal "error rate"
- Our expected error rate is at most optimal error rate + $\epsilon + \frac{\ln n}{\epsilon T}$
- Setting $\epsilon = \left(\frac{\ln n}{T}\right)^{1/2}$, our error rate $\leq$ optimal rate + 2 $\left(\frac{\ln n}{T}\right)^{1/2}$
- The last term is called the "regret". As T gets larger, the regret goes to 0