

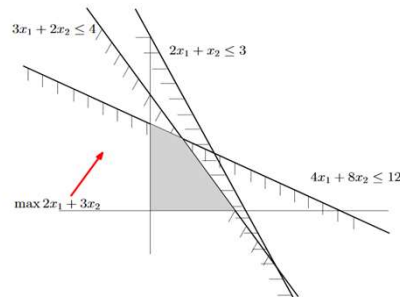
# Topic 9: Linear Programming III

David Woodruff

## Outline

- Linear Programming Duality
- Application to zero sum games

$$\begin{aligned}
 P &= \max(2x_1 + 3x_2) \\
 \text{s.t. } &4x_1 + 8x_2 \leq 12 \\
 &2x_1 + x_2 \leq 3 \\
 &3x_1 + 2x_2 \leq 4 \\
 &x_1, x_2 \geq 0
 \end{aligned}$$



Since  $2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12$ , we know  $\text{OPT} \leq 12$   
 Since  $2x_1 + 3x_2 \leq \frac{1}{2}(4x_1 + 8x_2) \leq 6$ , we know  $\text{OPT} \leq 6$   
 Since  $2x_1 + 3x_2 \leq \frac{1}{3}((4x_1 + 8x_2) + (2x_1 + x_2)) \leq 5$ , we know  $\text{OPT} \leq 5$

## Duality

- We took non-negative linear combinations of the constraints
- How do we find the best upper bound on OPT this way?
- Let  $y_1, y_2, y_3 \geq 0$  be the coefficients of our linear combination. Then,

$$4y_1 + 2y_2 + 3y_3 \geq 2$$

$$8y_1 + y_2 + 2y_3 \geq 3$$

$$y_1, y_2, y_3 \geq 0$$

and we seek  $\min(12y_1 + 3y_2 + 4y_3)$

$$P = \max(2x_1 + 3x_2)$$

$$\text{s.t. } 4x_1 + 8x_2 \leq 12$$

$$2x_1 + x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

## Primal LP

$$\begin{aligned}
 P &= \max(2x_1 + 3x_2) \\
 \text{s.t. } &4x_1 + 8x_2 \leq 12 \\
 &2x_1 + x_2 \leq 3 \\
 &3x_1 + 2x_2 \leq 4 \\
 &x_1, x_2 \geq 0
 \end{aligned}$$

## Dual LP

$$\begin{aligned}
 &4y_1 + 2y_2 + 3y_3 \geq 2 \\
 &8y_1 + y_2 + 2y_3 \geq 3 \\
 &y_1, y_2, y_3 \geq 0 \\
 \text{and we seek } &\min(12y_1 + 3y_2 + 4y_3)
 \end{aligned}$$

- If  $(x_1, x_2)$  is feasible for the primal, and  $(y_1, y_2, y_3)$  feasible for the dual,  
 $2x_1 + 3x_2 \leq 12y_1 + 3y_2 + 4y_3$
- If these are equal, we've found the optimal value for both LPs
- $(x_1, x_2) = (\frac{1}{2}, \frac{5}{4})$  and  $(y_1, y_2, y_3) = (\frac{5}{16}, 0, \frac{1}{4})$  give the same value 4.75, so optimal

## Dual LP

$$\begin{aligned}
 &4y_1 + 2y_2 + 3y_3 \geq 2 \\
 &8y_1 + y_2 + 2y_3 \geq 3 \\
 &y_1, y_2, y_3 \geq 0
 \end{aligned}$$

and we seek  $\min(12y_1 + 3y_2 + 4y_3)$

- Let's try do the same thing to the dual:
- $12y_1 + 3y_2 + 4y_3 \geq 4y_1 + 2y_2 + 3y_3 \geq 2$
- $12y_1 + 3y_2 + 4y_3 \geq 8y_1 + y_2 + 2y_3 \geq 3$
- $12y_1 + 3y_2 + 4y_3 \geq \frac{2}{3}(4y_1 + 2y_2 + 3y_3) + (8y_1 + y_2 + 2y_3) \geq \frac{4}{3} + 3$

Dual LP

$$\begin{aligned}
 &4y_1 + 2y_2 + 3y_3 \geq 2 \\
 &8y_1 + y_2 + 2y_3 \geq 3 \\
 &y_1, y_2, y_3 \geq 0 \\
 \text{and we seek } &\min(12y_1 + 3y_2 + 4y_3)
 \end{aligned}$$

$$\begin{aligned}
 P &= \max(2x_1 + 3x_2) \\
 \text{s.t. } &4x_1 + 8x_2 \leq 12 \\
 &2x_1 + x_2 \leq 3 \\
 &3x_1 + 2x_2 \leq 4 \\
 &x_1, x_2 \geq 0
 \end{aligned}$$

- Take non-negative linear combination of the two constraints
- How do we find the best lower bound on OPT this way?
- Let  $x_1, x_2 \geq 0$  be the coefficients of our linear combination. Then,
- $4x_1 + 8x_2 \leq 12$ ,  $2x_1 + x_2 \leq 3$ ,  $3x_1 + 2x_2 \leq 4$ ,  $x_1 \geq 0$ ,  $x_2 \geq 0$   
 and we seek to maximize  $2x_1 + 3x_2$

We got back the **primal!**

## Non-Nice Constraints

$$\begin{aligned}
 P &= \max(7x_1 - x_2 + 5x_3) \\
 \text{s.t. } &x_1 + x_2 + 4x_3 \leq 8 \\
 &3x_1 - x_2 + 2x_3 \geq 3 \\
 &x_1, x_2, x_3 \geq 0
 \end{aligned}$$

$$\begin{aligned}
 D &= \min(8y_1 + 3y_2) \\
 \text{s.t. } &y_1 + 3y_2 \geq 7 \\
 &y_1 - y_2 \geq -1 \\
 &4y_1 + 2y_2 \geq 5 \\
 &y_1 \geq 0, y_2 \leq 0
 \end{aligned}$$

## Formal Definition of Duality

### Primal

$$\begin{aligned} \text{Max } c^T x \\ \text{subject to } Ax \leq b \\ x \geq 0 \end{aligned}$$

### Dual

$$\begin{aligned} \text{Min } b^T y \\ \text{subject to } A^T y \geq c \\ y \geq 0 \end{aligned}$$

- Dual of the dual is the primal!
- Can we get better upper/lower bounds by looking at more complicated combinations of the inequalities, not just linear combinations?

## Weak Duality

### Primal

$$\begin{aligned} \text{Max } c^T x \\ \text{subject to } Ax \leq b \\ x \geq 0 \end{aligned}$$

### Dual

$$\begin{aligned} \text{Min } b^T y \\ \text{subject to } A^T y \geq c \\ y \geq 0 \end{aligned}$$

- (Weak Duality) If  $x$  is a feasible solution of the primal, and  $y$  is a feasible solution of the dual, then  $c^T x \leq b^T y$

- Proof: Since  $x \geq 0$  and  $y \geq 0$ ,  

$$c^T x \leq y^T A x \leq y^T b = b^T y$$

## Strong Duality

### Primal

$$\begin{aligned} \text{Max } c^T x \\ \text{subject to } Ax \leq b \\ x \geq 0 \end{aligned}$$

### Dual

$$\begin{aligned} \text{Min } b^T y \\ \text{subject to } A^T y \geq c \\ y \geq 0 \end{aligned}$$

- (Strong Duality) If primal is feasible and bounded (i.e., optimal value is not  $\infty$ ), then dual is feasible and bounded (and if dual is feasible and bounded, so is the primal). If  $x^*$  is optimal solution to the primal, and  $y^*$  is optimal solution to dual, then  

$$c^T x^* = b^T y^*$$
- To prove  $x^*$  is optimal, I can give you  $y^*$  and you can check if  $x^*$  is feasible for the primal,  $y^*$  is feasible for the dual, and  $c^T x^* = b^T y^*$

## Consequences of Duality

$P \backslash D$	$I$	$O$	$U$
$I$	?	?	?
$O$	?	?	?
$U$	?	?	?

$I$  means infeasible

$O$  means feasible and bounded

$U$  means unbounded

*Which combinations are possible?*

### Consequences of Duality

$P \backslash D$	$I$	$O$	$U$
$I$	✓	X	✓
$O$	X	✓	X
$U$	✓	X	X

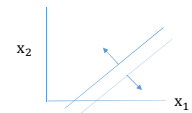
I means infeasible  
 O means feasible and bounded  
 U means unbounded

*Check means possible*  
*X means impossible*

### Possible Scenarios

$P \backslash D$	$I$	$O$	$U$
$I$	✓	X	✓
$O$	X	✓	X
$U$	✓	X	X

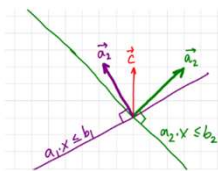
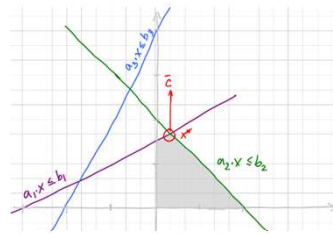
- Suppose primal is feasible and bounded
- By strong duality, dual is feasible and bounded
- If primal (maximization) is unbounded, by weak duality,  $c^T x \leq b^T y$ , so no feasible dual solution e.g.,  $\max x_1$  subject to  $x_1 \geq 1$  and  $x_1 \geq 0$   
 dual will have  $y_1 \leq 0$  and  $y_1 \geq 1$



- Can primal and dual both be infeasible?
- **Primal:**  $\max 2x_1 - x_2$  subject to  $x_1 - x_2 \leq 1$  and  $-x_1 + x_2 \leq -2$  and  $x_1 \geq 0, x_2 \geq 0$
- **Dual:**  $y_1 \geq 0, y_2 \geq 0$ , and  $y_1 - y_2 \geq 2$  and  $-y_1 + y_2 \geq -1$ , and  $\min y_1 - 2y_2$
- Constraints are same for primal and dual, and both infeasible

### Strong Duality Intuition

Suppose  $x^*$  satisfies  $a_1 x = b_1$  and  $a_2 x = b_2$



### Strong Duality Intuition

- For non-negative  $y_1$  and  $y_2$

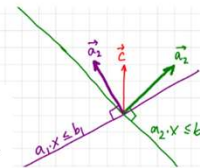
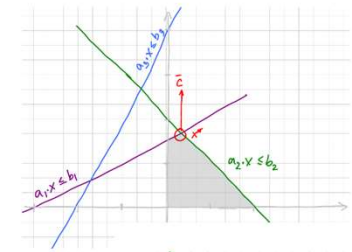
$$c = y_1 a_1 + y_2 a_2.$$

$$\begin{aligned} c^T x^* &= (y_1 a_1 + y_2 a_2) \cdot x^* \\ &= y_1 (a_1 \cdot x^*) + y_2 (a_2 \cdot x^*) \\ &= y_1 b_1 + y_2 b_2 \end{aligned}$$

Defining  $y = (y_1, y_2, 0, \dots, 0)$ , we get

$$\text{optimal value of primal} = c^T x^* = b^T y = \text{value of dual solution } y.$$

the  $y$  we found satisfies  $c = y_1 a_1 + y_2 a_2 = \sum_i y_i a_i = A^T y$ , and hence  $y$  satisfies the dual constraints  $y^T A \geq c^T$  by construction. But  $b^T y \geq c^T x^*$  by weak duality, so  $y$  is optimal!



## Duality in Zero-Sum Games

- R is an  $n \times m$  row payoff matrix
- W.l.o.g. R has all non-negative entries
- Variables:  $v, p_1, \dots, p_n$
- Max  $v$   
subject to  $p_i \geq 0$  for all rows  $i$ ,  $\sum_i p_i = 1$ ,  $\sum_i p_i R_{i,j} \geq v$  for all columns  $j$
- Replace  $\sum_i p_i = 1$  with  $\sum_i p_i \leq 1$ .
- Include  $v \geq 0$
- Write  $\sum_i p_i R_{i,j} \geq v$  as  $v - \sum_i p_i R_{i,j} \leq 0$

## Duality in Zero-Sum Games

max  $c^T x$  subject to  $Ax \leq b$  and  $x \geq 0$

$$x = \begin{bmatrix} v \\ p_1 \\ p_2 \\ \dots \\ p_n \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & & & & & & \\ 1 & & & & & & \\ \dots & & & & & & \\ 1 & & & & & & \\ 0 & 1 & \dots & 1 \end{bmatrix}.$$

- Dual: min  $y^T b$  subject to  $y^T A \geq c^T$  and  $y \geq 0$  for  $y = (y_1, \dots, y_{m+1})$
- Dual constraints say  $y_1 + \dots + y_m \geq 1$  and  $\sum_j y_j R_{ij} \leq y_{m+1}$  for all rows  $i$ 
  - Since we're minimizing  $y_{m+1}$  and  $R_{ij}$  all non-negative,  $y_1 + \dots + y_m = 1$
- $y_{m+1}$  is value to the row player and  $y_1, \dots, y_m$  is column player's strategy
- **Strong duality:**  $\max_p \min_j \sum_i p_i R_{ij} = \min_{y_1, \dots, y_m} \max_i \sum_j y_j R_{ij}$