Topic 4: The Data Stream Model

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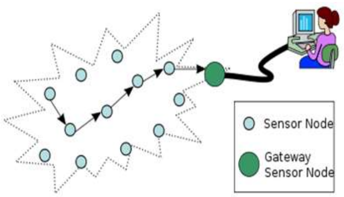
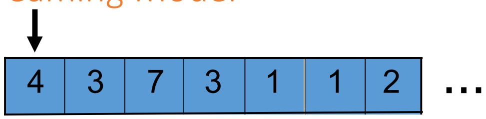
Data Streams

- A stream is a sequence of data, that is too large to be stored in available memory
- Examples
  - Internet search logs
  - Network Traffic
  - Sensor networks



  - Scientific data streams (astronomical, genomics, physical simulations)…

Streaming Model



- Stream of elements $a_1, \ldots, a_i, \ldots$ each from an alphabet Σ and taking b bits to represent
- Single or small number of passes over the data
- Almost all algorithms are randomized and approximate
  - Usually necessary to achieve efficiency
  - Randomness is in the algorithm, not the input
- Goals: minimize space complexity (in bits), processing time

Example Streaming Problems

- Let $a_{[1:t]} = \langle a_1, \ldots, a_t \rangle$ be the first t elements of the stream
- Suppose $a_1, \ldots, a_t$ are integers in $\{-2^b + 1, -2^b + 2, \ldots, -1, 0, 1, 2, \ldots, 2^b - 1\}$
  - Example stream: 3, 1, 17, 4, -9, 32, 101, 3, -722, 3, 900, 4, 32
- How many bits do we need to maintain $f(a_{[1:t]}) = \sum_{i=1,\ldots,t} a_i$?
  - Outputs on example: 3, 4, 21, 25, 16, 48, 149, 152, -570, -567, 333, 337, 379, …
  - O(b + log t)
- How many bits do we need to maintain $f(a_{[1:t]}) = \max_{i=1,\ldots,t} a_i$?
  - Outputs on example: 3, 3, 17, 17, 17, 32, 101, 101, 101, 101, 900, 900, 900, …
  - O(b) bits

Example Streaming Problems

- The median of all the numbers we've stored so far
  - Example stream: 3, 1, 17, 4, -9, 32, 101, 3, -722, 3, 900, 4, 32
  - Median: 3, 1, 3, 3, 3, 3, 4, 3, …
  - This seems harder…
- The number of distinct elements we've seen so far?
  - Outputs on example: 1, 2, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9, 9, …
- The elements that have appeared at least an ϵ-fraction of the time? These are the ϵ-heavy hitters
  - Cover today

Many Applications

- Internet router may want to figure out which IP connections are heavy hitters, e.g., the ones that use more than .01% of your bandwidth
- Or maybe the router wants to know the median (or 90-th percentile) of the file sizes being transferred
- Hashing is a key technique

Finding ϵ-Heavy Hitters

- S_t is the multiset of items at time t, so $S_0 = \emptyset$, $S_1 = \{a_1\}$, …, $S_i = \{a_1, \ldots, a_i\}$,
  $\text{count}_t(e) = |\{i \in \{1, 2, \ldots, t\} \text{ such that } a_i = e\}|$
- $e \in \Sigma$ is an ϵ-heavy hitter at time t if $\text{count}_t(e) > \epsilon \cdot t$
- Given $\epsilon > 0$, can we output the ϵ-heavy hitters?
  - Let's output a set of size $\frac{1}{\epsilon}$ containing all the ϵ-heavy hitters
- Note: can output "false positives" but not allowed to output "false negatives", i.e., not allowed to miss any heavy hitter, but could output non-heavy hitters

Finding ϵ-Heavy Hitters

- Example: E, D, B, D, D_5 D, B, A, C, B_{10} B, E, E, E, E_{15}, E
  (the subscripts are just to help you count)
- At time 5, the element D is the only 1/3-heavy hitter
- At time 11, both B and D are 1/3-heavy hitters
- At time 15, there is no 1/3-heavy hitter
- At time 16, only E is a 1/3-heavy hitter

Can't afford to keep counts of all items, so how to maintain a short summary to output the ϵ-heavy hitters?

Finding a Majority Element

- First find a .5-heavy hitter, that is, a majority element:
  memory ← empty and counter ← 0
  when element a_t arrives
    if (counter == 0)
      memory ← a_t and counter ← 1
    else
      if a_t = memory
        counter ++
      else
        counter - -
        (discard a_t)
- At end of the stream, return the element in memory

$\cancel{3}\ \cancel{1}\ \cancel{2}\ \cancel{1}\ 1$

Memory = 3, Count = 1
Memory = 3, Count = 0
Memory = 2, Count = 1
Memory = 2, Count = 0
Memory = 1, Count = 1

Analysis of Finding a Majority Element

- If there is no majority element, we output a false positive, which is OK
- If there is a majority element, we will output it. Why?
  - When we discard an element a_t, we throw away a different element
  - Every time we throw away a copy of a majority element, we throw away another element, but majority element is more than half the total number of elements, so can't throw away all of them

Extending to ϵ-Heavy Hitters

Set $k = \left\lceil \frac{1}{\epsilon} \right\rceil - 1$
Array T[1, …, k], where each location can hold one element from Σ
Array C[1, …, k], where each location can hold a non-negative integer
C[i] ← 0 and T[i] ←⊥ **for** all i

If there is j $\in \{1, 2, \ldots, k\}$ such that $a_t = T[j]$, **then** $C[j]++$
Else if some counter C[j] = 0 **then** $T[j] \leftarrow a_t$ and C[j] ← 1
Else decrement all counters by 1 (and discard element a_t)

$est_t(e) = C[j]$ if e == $T[j]$ for some j, and $est_t(e) = 0$ otherwise

Analyzing Counts

- Lemma: $0 \le \mathrm{count}_t(e) - \mathrm{est}_t(e) \le \frac{t}{k+1} \le \epsilon \cdot t$
- Proof: $\mathrm{count}_t(e) \ge \mathrm{est}_t(e)$ since we never increase a counter for e unless we see e

If we don't increase $\mathrm{est}_t(e)$ by 1 when we see an update to e, we decrement k counters and discard the current update to e

So we drop k+1 distinct stream updates, but there are t total updates, so we won't increase $\mathrm{est}_t(e)$ by 1, when we should, at most $\frac{t}{k+1} \le \epsilon \cdot t$ times

Heavy Hitters Guarantee

- At any time t, all ϵ-heavy hitters e are in the array T. Why?
- For an ϵ-heavy hitter e, we have $\mathrm{count}_t(e) > \epsilon \cdot t$
- But $\mathrm{est}_t(e) \ge \mathrm{count}_t(e) - \epsilon \cdot t$
- So $\mathrm{est}_t(e) > 0$, so e is in array T
- Space is O(k (log(Σ) + log t)) = O(1/ ϵ) (log(Σ) + log t) bits

Heavy Hitters with Deletions

- Suppose we can delete elements e that have already appeared
- Example: (add, A), (add, B), (add, A), (del, B), (del, A), (add, C)
- Multisets at different times
  $S_0 = \emptyset, S_1 = \{A\}, S_2 = \{A, B\}, S_3 = \{A, A, B\}, S_4 = \{A, A\}, S_5 = \{A\}, S_6 = \{A, C\}, \ldots$
- "active" set S_t has size $|S_t| = \sum_{e \in \Sigma} \mathrm{count}_t(e)$ and can grow and shrink

Data Structure for Approximate Counts

- Query "What is $\mathrm{count}_t(e)$?", should output $\mathrm{est}_t(e)$ with:
  $$\Pr[|\mathrm{est}_t(e) - \mathrm{count}_t(e)| \le \epsilon |S_t|] \ge 1 - \delta$$
- Want space close to our previous O(1/ ϵ) (log(Σ) + log t) bits
- Let $h: \Sigma \to \{0,1,2,\ldots,k-1\}$ be a hash function (will specify later)
- Maintain an array A[0, 1, …, k-1] to store non-negative integers

when update a_t arrives:
  if $a_t = (\mathrm{add}, e)$ **then** $A[h(e)]++$
  else $a_t = (\mathrm{del}, e)$, and $A[h(e)]--$

- $\mathrm{est}_t(e) = A[h(e)]$

Data Structure for Approximate Counts

- $A[h(e)] = \sum_{e' \in \Sigma} \text{count}_t(e') \cdot \mathbf{1}(h(e') = h(e))$, where $\mathbf{1}$(condition) evaluates to 1 if the condition is true, and evaluates to 0 otherwise
- $A[h(e)] = \text{count}_t(e) + \sum_{e' \neq e} \text{count}_t(e') \cdot \mathbf{1}(h(e') = h(e))$,
- $\text{est}_t(e) - \text{count}_t(e) = \sum_{e' \neq e} \text{count}_t(e') \cdot \mathbf{1}(h(e') = h(e))$
- Since we have a small array A with k locations, there are likely many $e' \neq e$ with $h(e') = h(e)$, but can we bound the expected error?

Data Structure for Approximate Counts

- Recall: Family H of hash functions h: U -> {0, 1, …, k-1} is universal if for all $x \neq y$,

$$\Pr_{h \leftarrow H}[h(x) = h(y)] \leq \frac{1}{k}$$

- Gave a simple family where h can be specified using O(log |U|) bits. Here, |U| = |Σ|
- $E[\text{est}_t(e) - \text{count}_t(e)] = E[\sum_{e' \neq e} \text{count}_t(e') \cdot \mathbf{1}(h(e') = h(e))]$

$$= \sum_{e' \neq e} \text{count}_t(e') \cdot E[\mathbf{1}(h(e') = h(e))]$$
$$= \sum_{e' \neq e} \text{count}_t(e') \cdot \Pr[h(e') = h(e)]$$
$$\leq \sum_{e' \neq e} \text{count}_t(e') \cdot \left(\frac{1}{k}\right)$$
$$= \frac{|S_t| - \text{count}_t(e)}{k} \leq \frac{|S_t|}{k}$$

$k = 1/\epsilon$ makes this at most $\epsilon \cdot |S_t|$. Space is $O(\frac{1}{\epsilon})$ counters plus storing hash function

High Probability Bounds for CountMin

- Have $0 \leq \text{est}_t(e) - \text{count}_t(e) \leq |S_t|/k$ in expectation from CountMin
  - With probability 1/2, $\text{est}_t(e) - \text{count}_t(e) \leq 2|S_t|/k$ Why?
- Can we make the success probability 1-δ?
  - Independent repetition: pick m hash functions $h_1, \ldots, h_m$ with $h_i: \Sigma \to \{0, 1, 2, \ldots, k-1\}$ independently from H. Create array A_i for h_i

    when update a_t arrives:

    for each i from 1 to m

    if $a_t = (\text{add}, e)$ **then** $A_i[h_i(e)]++$

    else $a_t = (\text{del}, e)$ and $A_i[h_i(e)]--$

High Probability Bounds and Overall Space

What is our new estimate of $\text{count}_t(e)$?

$$\texttt{best}_t(e) := \min_{i=1}^{m} A_i[h_i(e)].$$

- Each $A_i[h_i(e)]$ is an *overestimate* to $\text{count}_t(e)$
- By independence, $\Pr[\text{for all } i,\ A_i[h_i(e)] \geq 2|S_t|/k] \leq \left(\frac{1}{2}\right)^m$
- For $k = \frac{2}{\epsilon}$ and $m = \log_2\left(\frac{1}{\delta}\right)$, the error is at most $\epsilon|S_t|$ with probability 1-δ
- Space: $m \cdot k = O(\frac{\log(\frac{1}{\delta})}{\epsilon})$ counters each of O(lg t) bits

  $m \cdot O(\log|\Sigma|) = O(\log\left(\frac{1}{\delta}\right)\log|\Sigma|)$ bits to store hash functions

ϵ-Heavy Hitters

- Our new estimate $\text{best}_t(e)$ satisfies

$$\Pr[|\text{best}_t(e) - \text{count}_t(e)| \leq \epsilon|S_t|] \geq 1 - \delta$$

and uses $O(\frac{\log\left(\frac{1}{\delta}\right)\log t}{\epsilon} + \log\left(\frac{1}{\delta}\right)\log|\Sigma|)$ bits of space

- What if we want with probability 9/10, simultaneously for all e, $|\text{best}_t(e) - \text{count}_t(e)| \leq \epsilon|S_t|$?
- Set $\delta = \frac{1}{10|\Sigma|}$ and apply a union bound over all $e \in \Sigma$