Topic 1: Introduction and Median Finding

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Course homepage:

http://www.cs.cmu.edu/~dwoodruf/teaching/15451-sum19/cis.html

Grading and Course Policies

3 Written Homeworks	**30% (10% each)**
Class Participation	**10%**
1 Exam (in class)	**20%**
Research project	**40%**

Schedule of Lectures and Exams

- Daily in A3-301
  - lecture 8:15-10:05, A class lecture
  - lecture 10:15-12:15, B class TA session
  - lecture 13:30-15:20, B class lecture
  - Lecture 15:30-17:30, A class TA session
- First two weeks the lectures will cover theoretical background
- Last two weeks the lectures will be project-oriented
- There is no class on 8/7, I will be away. We will schedule extra lectures/offices hours to make up for it
- Exam: during lecture 7/26

Homework

- HW1: out 7/15, due 7/18 at 8:15am
- HW2: out 7/18, due 7/22 at 8:15am
- HW3: out 7/22, due 7/25 at 8:15am
- For the homework, you will be asked to design and/or analyze algorithms. Your solution should be written up formally – that is, you should prove your claims.
- You can work by yourself or with at most one other person - you must list your collaborator (if you have one) and write the solutions yourself. You're allowed to read additional textbooks or online notes but must cite them. In particular this is useful:

http://www.cs.cmu.edu/~15451/policies.html

Schedule of Topics

- 7/15: median-finding and concrete upper and lower bounds
- 7/16: concrete upper and lower bounds and hashing
- 7/17: streaming and fingerprinting
- 7/18: fingerprinting and game theory
- 7/19: linear programming basics and algorithms

- 7/22: linear programming algorithms and duality
- 7/23: graph compression and sketching
- 7/24: sketching and graph streaming algorithms
- 7/25: graph streaming algorithms and nearest neighbor search
- 7/26: linear algebra review and regression
- 7/26: review and Exam

Goals of the Course

- Design and analyze algorithms!
- Algorithms: dynamic programming, divide-and-conquer, hashing and data structures, randomization, network flows, linear programming
- Analysis: recurrences, probabilistic analysis, amortized analysis, potential functions
- Dual to Algorithms: complexity theory and lower bounds
- New Models: online algorithms, machine learning, data streams

Guarantees on Algorithms

- Want provable guarantees on the running time of algorithms
- Why?
- Composability: if we know an algorithm runs in time at most T on any input, don't have to worry what kinds of inputs we run it on
- Scaling: how does the time grow as the input size grows?
- Designing better algorithms: what are the most time-consuming steps?

Example: Median Finding

- In the median-finding problem, we have an array

$$a_1, a_2, \ldots, a_n$$

and want the index i for which there are exactly $\lfloor n/2 \rfloor$ numbers larger than a_i

- *How can we find the median?*
  - Check each item to see if it is the median: $\Theta(n^2)$ time
  - Sort items with MergeSort (deterministic) or QuickSort (randomized): $\Theta(n \log n)$ time
  - Can we find it faster? What about finding the k-th smallest number?

QuickSelect Algorithm to Find the k-th Smallest Number

- Assume $a_1, a_2, \ldots, a_n$ are all distinct for simplicity
- Choose a random element a_i in the list – call this the "pivot"
- Compare each a_j to a_i
  - Let LESS = {a_j such that $a_j < a_i$}
  - Let GREATER = {a_j such that $a_j > a_i$}
- If $k \leq |\text{LESS}|$, find the k-th smallest element in LESS
- If $k = |\text{LESS}| + 1$, output the pivot a_i
- Else find the (k-|LESS|-1)-th smallest item in GREATER
- Similar to Randomized QuickSort, but *only recurse on one side!*

Bounding the Running Time

- Theorem: the expected number of comparisons for QuickSelect is at most 4n
- Let $T(n) = \max_k T(n,k)$, where T(n,k) is the expected number of comparisons to find the k-th smallest item in an array of length n, maximized over all arrays
- T(n) is a non-decreasing function of n
- Let's show T(n) < 4n by induction
- Base case: T(1) = 0 < 4
- Inductive hypothesis: T(n-1) < 4(n-1)

Bounding the Running Time

- Suppose we have an array of length n
- Pivot randomly partitions the array into two pieces, LESS and GREATER, with |LESS| + |GREATER| = n-1
  - |LESS| is uniform in the set {0, 1, 2, 3, …, n-1}
  - Since $T(i)$ is non-decreasing with i, to upper bound T(n) we can assume we recurse on larger half
  - $T(n) \leq n - 1 + \frac{2}{n}\sum_{i=\frac{n}{2},\ldots,n-1} T(i)$

$$\leq n - 1 + \frac{2}{n}\sum_{i=\frac{n}{2},\ldots,n-1} 4i \qquad \text{by inductive hypothesis}$$

$$< n - 1 + 4\left(\frac{3n}{4}\right) \qquad \text{since the average } \frac{2}{n}\sum_{i=\frac{n}{2},\ldots,n-1} i \text{ is at most } 3n/4$$

$$< 4n \qquad \text{completing the induction}$$

What About Deterministic Algorithms?

- Can we get an algorithm which does not use randomness and always performs O(n) comparisons?
- Idea: suppose we could deterministically find a pivot which partitions the input into two pieces LESS and GREATER each of size $\lfloor \frac{n}{2} \rfloor$
- How to do that?
- Find the median and then partition around that
  - Um… finding the median is the original problem we want to solve….

Deterministically Finding a Pivot

- Idea: deterministically find a pivot with O(n) comparisons to partition the input into two pieces LESS and GREATER each of size at least 3n/10
- DeterministicSelect:
  1. Group the array into n/5 groups of size 5 and find the median of each group
  2. Recursively, find the median of medians. Call this p
  3. Use p as a pivot to split into subarrays LESS and GREATER
  4. Recurse on the appropriate piece
- Theorem: DeterministicSelect makes O(n) comparisons to find the k-th smallest item in an array of size n

Running Time of DeterministicSelect

- DeterministicSelect:
  1. Group the array into n/5 groups of size 5 and find the median of each group
  2. Recursively, find the median of medians. Call this p
  3. Use p as a pivot to split into subarrays LESS and GREATER
  4. Recurse on the appropriate piece
- Step 1 takes O(n) time since it takes O(1) time to find the median of 5 elements
- Step 2 takes T(n/5) time
- Step 3 takes O(n) time

Claim: $|\text{LESS}| \geq 3n/10\text{-}1$ and $|\text{GREATER}| \geq 3n/10\text{-}1$

Running Time of DeterministicSelect

- Claim: $|\text{LESS}| \geq 3n/10\text{-}1$ and $|\text{GREATER}| \geq 3n/10\text{-}1$
- **Example 1:** If n = 15, we have three groups of 5:

  {1, 2, 3, 10, 11}, {4, 5, 6, 12, 13}, {7,8,9,14,15}

  medians: 3 6 9

  median of medians p: 6
- There are g = n/5 groups, and at least $\lceil \frac{g}{2} \rceil$ of them have at least 3 elements at most p. The number of elements less than or equal to p is at least

$$3\left\lceil \frac{g}{2} \right\rceil \geq \frac{3n}{10}$$

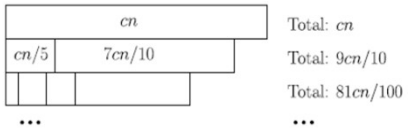
- Also at least 3n/10 elements greater than or equal to p

Running Time of DeterministicSelect

- DeterministicSelect:
  1. Group the array into n/5 groups of size 5 and find the median of each group
  2. Recursively, find the median of medians. Call this p
  3. Use p as a pivot to split into subarrays LESS and GREATER
  4. Recurse on the appropriate piece
- Steps 1-3 take O(n) + T(n/5) time
- Since $|\text{LESS}| \geq 3n/10\text{-}1$ and $|\text{GREATER}| \geq 3n/10\text{-}1$, Step 4 takes at most T(7n/10) time
- So $T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$, for a constant c > 0

Running Time of DeterministicSelect

- $T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$



- Time is $cn\left(1 + \left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)^2 + \ldots\right) \leq 10cn$
- Recurrence works because n/5 + 7n/10 < n

- For constants c and $a_1, a_2, \ldots a_r$ with $a_1 + a_2 + \cdots a_r < 1$, the recurrence $T(n) \leq T(a_1 n) + T(a_2 n) + \ldots + T(a_r n) + cn$ solves to $T(n) = O(n)$
  - If instead $a_1 + a_2 + \ldots + a_r = 1$, the recurrence solves to T(n) = O(n log n)
  - If we use median of 3 in DeterministicSelect instead of median of 5, what happens?