Topic 1: Introduction and Median Finding

David Woodruff

Course homepage:
http://www.cs.cmu.edu/~dwoodruf/teaching/15451-spr20/cis.html

Grading and Course Policies

3 Written Homeworks	**30% (10% each)**
Class Participation	**10%**
1 Exam (in class)	**20%**
Research project	**40%**

Schedule of Lectures and Exams

- Saturdays 7-10pm ET
  - On zoom
- First four weeks the lectures will cover theoretical background
- Remaining weeks the lectures will be project-oriented
- Exam: second half of lecture on 4/11

Homework

- HW1: out 3/21, due 3/28 at 11:59pm China time
- HW2: out 3/28, due at 4/4 at 11:59pm China time
- HW3: out 4/4, due 4/11 at 11:59pm China time
- For the homework, you will be asked to design and/or analyze algorithms. Your solution should be written up formally – that is, you should prove your claims.
- You can work by yourself or with at most one other person - you must list your collaborator (if you have one) and write the solutions yourself. You're allowed to read additional textbooks or online notes but must cite them.

Schedule of Topics

- First class: median-finding and concrete upper and lower bounds
- Second class: concrete lower bounds and hashing
- Third class: streaming and fingerprinting
- Fourth class: graph compression and Exam

Goals of the Course

- Design and analyze algorithms!
- Algorithms: dynamic programming, divide-and-conquer, hashing and data structures, randomization, network flows, linear programming
- Analysis: recurrences, probabilistic analysis, amortized analysis, potential functions
- Dual to Algorithms: complexity theory and lower bounds
- New Models: online algorithms, machine learning, data streams

Guarantees on Algorithms

- Want provable guarantees on the running time of algorithms
- Why?
- Composability: if we know an algorithm runs in time at most T on any input, don't have to worry what kinds of inputs we run it on
- Scaling: how does the time grow as the input size grows?
- Designing better algorithms: what are the most time-consuming steps?

Example: Median Finding

- In the median-finding problem, we have an array

$$a_1, a_2, \dots, a_n$$

and want the index i for which there are exactly $\lfloor n/2 \rfloor$ numbers larger than a_i

- *How can we find the median?*
  - Check each item to see if it is the median: $\Theta(n^2)$ time
  - Sort items with MergeSort (deterministic) or QuickSort (randomized): $\Theta(n \log n)$ time
  - Can we find it faster? What about finding the k-th smallest number?

QuickSelect Algorithm to Find the k-th Smallest Number

- Assume $a_1, a_2, \ldots, a_n$ are all distinct for simplicity
- Choose a random element a_i in the list – call this the "pivot"
- Compare each a_j to a_i
  - Let LESS = $\{a_j$ such that $a_j < a_i\}$
  - Let GREATER = $\{a_j$ such that $a_j > a_i\}$
- If $k \leq |\text{LESS}|$, find the k-th smallest element in LESS
- If $k = |\text{LESS}| + 1$, output the pivot a_i
- Else find the (k-|LESS|-1)-th smallest item in GREATER
- Similar to Randomized QuickSort, but *only recurse on one side!*

Bounding the Running Time

- Theorem: the expected number of comparisons for QuickSelect is at most 4n
- Let $T(n) = \max_k T(n, k)$, where T(n,k) is the expected number of comparisons to find the k-th smallest item in an array of length n, maximized over all arrays
- T(n) is a non-decreasing function of n
- Let's show T(n) < 4n by induction
- Base case: T(1) = 0 < 4
- Inductive hypothesis: T(n-1) < 4(n-1)

Bounding the Running Time

- Suppose we have an array of length n
- Pivot randomly partitions the array into two pieces, LESS and GREATER, with |LESS| + |GREATER| = n-1
  - |LESS| is uniform in the set {0, 1, 2, 3, …, n-1}
  - Since $T(i)$ is non-decreasing with i, to upper bound T(n) we can assume we recurse on larger half
  - $T(n) \leq n - 1 + \frac{2}{n} \sum_{i=\frac{n}{2},\ldots,n-1} T(i)$

$\leq n - 1 + \frac{2}{n} \sum_{i=\frac{n}{2},\ldots,n-1} 4i$ by inductive hypothesis

$< n - 1 + 4\left(\frac{3n}{4}\right)$ since the average $\frac{2}{n}\sum_{i=\frac{n}{2},\ldots,n-1} i$ is at most 3n/4

$< 4n$ completing the induction

What About Deterministic Algorithms?

- Can we get an algorithm which does not use randomness and always performs O(n) comparisons?
- Idea: suppose we could deterministically find a pivot which partitions the input into two pieces LESS and GREATER each of size $\lfloor \frac{n}{2} \rfloor$
- How to do that?
- Find the median and then partition around that
  - Um… finding the median is the original problem we want to solve….

Deterministically Finding a Pivot

- Idea: deterministically find a pivot with O(n) comparisons to partition the input into two pieces LESS and GREATER each of size at least 3n/10-1
- DeterministicSelect:
  1. Group the array into n/5 groups of size 5 and find the median of each group
  2. Recursively, find the median of medians. Call this p
  3. Use p as a pivot to split into subarrays LESS and GREATER
  4. Recurse on the appropriate piece
- Theorem: DeterministicSelect makes O(n) comparisons to find the k-th smallest item in an array of size n

Running Time of DeterministicSelect

- DeterministicSelect:
  1. Group the array into n/5 groups of size 5 and find the median of each group
  2. Recursively, find the median of medians. Call this p
  3. Use p as a pivot to split into subarrays LESS and GREATER
  4. Recurse on the appropriate piece
- Step 1 takes O(n) time since it takes O(1) time to find the median of 5 elements
- Step 2 takes T(n/5) time
- Step 3 takes O(n) time

Claim: |LESS| $\geq 3n/10$-1 and |GREATER| $\geq 3n/10$-1

Running Time of DeterministicSelect

- Claim: |LESS| $\geq 3n/10$-1 and |GREATER| $\geq 3n/10$-1
- **Example 1:** If n = 15, we have three groups of 5:

  {1, 2, 3, 10, 11}, {4, 5, 6, 12, 13}, {7,8,9,14,15}

  medians: 3 6 9

  median of medians p: 6
- There are g = n/5 groups, and at least $\lceil \frac{g}{2} \rceil$ of them have at least 3 elements at most p. The number of elements less than or equal to p is at least

$$3\left\lceil \frac{g}{2} \right\rceil \geq \frac{3n}{10}$$

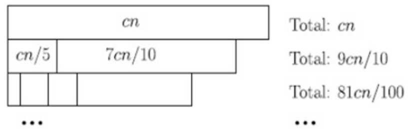
- Also at least 3n/10 elements greater than or equal to p

Running Time of DeterministicSelect

- DeterministicSelect:
  1. Group the array into n/5 groups of size 5 and find the median of each group
  2. Recursively, find the median of medians. Call this p
  3. Use p as a pivot to split into subarrays LESS and GREATER
  4. Recurse on the appropriate piece
- Steps 1-3 take O(n) + T(n/5) time
- Since |LESS| $\geq 3n/10$-1 and |GREATER| $\geq 3n/10$-1, Step 4 takes at most T(7n/10) time
- So $T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$, for a constant c > 0

Running Time of DeterministicSelect

- $T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$



- Time is $cn\left(1 + \left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)^2 + \ \ldots\right) \leq 10cn$
- Recurrence works because n/5 + 7n/10 < n

- For constants c and $a_1, a_2, \ldots a_r$ with $a_1 + a_2 + \cdots a_r < 1$, the recurrence $T(n) \leq T(a_1 n) + T(a_2 n) + \ \ldots + T(a_r n) + cn$ solves to $T(n) = O(n)$
  - If instead $a_1 + a_2 + \ \ldots + a_r = 1$, the recurrence solves to T(n) = O(n log n)
  - If we use median of 3 in DeterministicSelect instead of median of 5, what happens?