Algorithms, Spring 2020 at CIS

Homework 3

1. Use Strassen's algorithm to compute the matrix product $[1, 3; 7, 5] \cdot [6, 8; 4, 2]$. Show the intermediate quantities in Strassen's algorithm and the final result.

2. A family $\mathcal{H}$ of hash function $h : \{0, 1, \ldots, U\} \to \{0, 1, 2, \ldots, M - 1\}$ is a 2-universal family if for all pairs of distinct keys $x \neq y$, and all $z_1, z_2 \in \{0, 1, 2, \ldots, M - 1\}$, $\mathbf{Pr}_h[h(x) = z_1 \wedge h(y) = z_2] = 1/M^2$. Show that a 2-universal family $\mathcal{H}$ is also a universal family. Also, give an example of a universal family that is not 2-universal.

3. Suppose you are given an unweighted, undirected graph G on n vertices. Recall in class we showed the existence of a 2-additive spanner H with $O(n^{3/2})$ edges. The graph H was a subgraph of G and had the property that for every pair of vertices u and v in G, $d_G(u, v) \leq d_H(u, v) \leq d_G(u, v) + 2$.

   Now suppose we do not require that H be a subgraph, and further, we allow the edges of H to be weighted. Show how, with probability $1 - 1/n$, to find such an H with $O(n^{4/3} \log^2 n)$ edges such that for every pair of vertices u and v in G, $d_G(u, v) \leq d_H(u, v) \leq d_G(u, v) + 4$. Note that the distance in H is a weighted distance, that is, it is the sum of weights along the path from u to v in H with the minimum possible sum of weights.

   Hint: recall that in our 2-additive spanner construction, we separated edges based on those which had one endpoint of degree at most $n^{1/2}$, and those with both endpoints of degree larger than $n^{1/2}$. Here, consider separating edges based on those which have one endpoint of degree at most $n^{1/3}$, and those with both endpoints of degree larger than $n^{1/3}$. You will need to change the algorithm in the case that both enedpoints have degree larger than $n^{1/3}$. Think about sampling a random set of $O(n^{2/3} \log n)$ vertices and arguing similarly to the 2-additive construction, but use the fact that H is allowed to be weighted.