15-859(B) Machine Learning Theory

Game Theory

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Plan for Today

- 2-player zero-sum games
- 2-player general-sum games
- Many-player games with structure: congestion games / exact potential games
  - Best-response dynamics
  - Price of anarchy, Price of stability

2-Player Zero-Sum games

- Two players R and C. Zero-sum means that what's good for one is bad for the other.
- Game defined by matrix with a row for each of R's options and a column for each of C's options. Matrix tells who wins how much.
  - an entry (x,y) means: x = payoff to row player, y = payoff to column player. "Zero sum" means that y = -x.
- E.g., penalty shot:

goalie → columns; shooter → rows

	Left	Right
Left	(0,0)	(1,-1)
Right	(1,-1)	(0,0)

(1,-1): GOAALLL!!! ; (0,0): No goal

Game Theory terminolgy

- Rows and columns are called pure strategies.
- Randomized algs called mixed strategies.
- "Zero sum" means that game is purely competitive. (x,y) satisfies x+y=0. (Game doesn't have to be fair).

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Minimax-optimal strategies

- Minimax optimal strategy is a (randomized) strategy that has the best guarantee on its expected gain, over choices of the opponent. [maximizes the minimum]
- I.e., the thing to play if your opponent knows you well.

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Minimax-optimal strategies

- Can solve for minimax-optimal strategies using Linear programming
- No-regret strategies will do nearly as well or better.
- I.e., the thing to play if your opponent knows you well.

goalie → columns; shooter → rows

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Minimax Theorem (von Neumann 1928)

- Every 2-player zero-sum game has a unique value V.
- Minimax optimal strategy for R guarantees R's expected gain at least V.
- Minimax optimal strategy for C guarantees C's expected loss at most V.

Existence of no-regret strategies gives one way of proving the theorem.

Can use notion of minimax optimality to explain bluffing in poker

Simplified Poker (Kuhn 1950)

- Two players A and B.
- Deck of 3 cards: 1,2,3.
- Players ante $1.
- Each player gets one card.
- A goes first. Can bet $1 or pass.
  - If A bets, B can call or fold.
  - If A passes, B can bet $1 or pass.
    - If B bets, A can call or fold.
- High card wins (if no folding). Max pot $2.

- Two players A and B. 3 cards: 1,2,3.
- Players ante $1. Each player gets one card.
- A goes first. Can bet $1 or pass.
  - If A bets, B can call or fold.
  - If A passes, B can bet $1 or pass.
    - If B bets, A can call or fold.

Writing as a Matrix Game

- For a given card, A can decide to
  - Pass but fold if B bets. [PassFold]
  - Pass but call if B bets. [PassCall]
  - Bet. [Bet]
- Similar set of choices for B.

Can look at all strategies as a big matrix…

	[FP,FP,CB]	[FP,CP,CB]	[FB,FP,CB]	[FB,CP,CB]
[PF,PF,PC]	0	0	-1/6	-1/6
[PF,PF,B]	0	1/6	-1/3	-1/6
[PF,PC,PC]	-1/6	0	0	1/6
[PF,PC,B]	-1/6	-1/6	1/6	1/6
[B,PF,PC]	-1/6	0	0	1/6
[B,PF,B]	1/6	–1/3	0	–1/2
[B,PC,PC]	1/6	–1/6	–1/6	–1/2
[B,PC,B]	0	–1/2	1/3	–1/6
	0	–1/3	1/6	–1/6

And the minimax optimal strategies are…

- A:
  - If hold 1, then 5/6 PassFold and 1/6 Bet.
  - If hold 2, then ½ PassFold and ½ PassCall.
  - If hold 3, then ½ PassCall and ½ Bet.

  Has both bluffing and underbidding…
- B:
  - If hold 1, then 2/3 FoldPass and 1/3 FoldBet.
  - If hold 2, then 2/3 FoldPass and 1/3 CallPass.
  - If hold 3, then CallBet

  Minimax value of game is –1/18 to A.

Now, to General-Sum games…

General-sum games

- In general-sum games, can get win-win and lose-lose situations.
- E.g., "what side of sidewalk to walk on?":

person walking towards you

you

	Left	Right
Left	(1,1)	(-1,-1)
Right	(-1,-1)	(1,1)

General-sum games

- In general-sum games, can get win-win and lose-lose situations.
- E.g., "which movie should we go to?":

	Titans	Date night
Titans	(8,2)	(0,0)
Date night	(0,0)	(2,8)

No longer a unique "value" to the game.

Nash Equilibrium

- A Nash Equilibrium is a stable pair of strategies (could be randomized).
- Stable means that neither player has incentive to deviate on their own.
- E.g., "what side of sidewalk to walk on":

	Left	Right
Left	(1,1)	(-1,-1)
Right	(-1,-1)	(1,1)

NE are: both left, both right, or both 50/50.

Uses

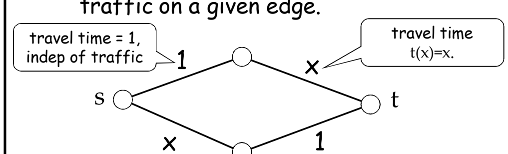
- Economists use games and equilibria as models of interaction.
- E.g., pollution / prisoner's dilemma:
  - (imagine pollution controls cost $4 but improve everyone's environment by $3)

	don't pollute	pollute
don't pollute	(2,2)	(-1,3)
pollute	(3,-1)	(0,0)

Need to add extra incentives to get good overall behavior.

NE can do strange things

- Braess paradox:
  - Road network, traffic going from s to t.
  - travel time as function of fraction x of traffic on a given edge.



Fine. NE is 50/50. Travel time = 1.5

NE can do strange things

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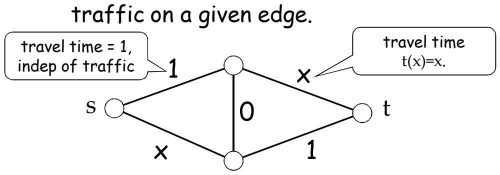

Add new superhighway. NE: everyone uses zig-zag path. Travel time = 2.

Existence of NE

- Nash (1950) proved: any general-sum game must have at least one such equilibrium.
  - Might require mixed strategies.
- This also yields minimax thm as a corollary.
  - Pick some NE and let V = value to row player in that equilibrium.
  - Since it's a NE, neither player can do better even knowing the (randomized) strategy their opponent is playing.
  - So, they're each playing minimax optimal.

Existence of NE in 2-player games

- Proof will be non-constructive.
- Unlike case of zero-sum games, we **do not know any** polynomial-time algorithm for finding Nash Equilibria in $n \times n$ general-sum games. [known to be "PPAD-hard"]
- Notation:
  - Assume an nxn matrix.
  - Use $(p_1,\ldots,p_n)$ to denote mixed strategy for row player, and $(q_1,\ldots,q_n)$ to denote mixed strategy for column player.

Proof

- We'll start with Brouwer's fixed point theorem.
  - Let S be a compact convex region in R^n and let $f:S \rightarrow S$ be a continuous function.
  - Then there must exist $x \in S$ such that $f(x)=x$.
  - x is called a "fixed point" of f.
- Simple case: S is the interval [0,1].
- We will care about:
  - S = {(p,q): p,q are legal probability distributions on 1,...,n}. I.e., $S = \text{simplex}_n \times \text{simplex}_n$

Proof (cont)

- S = {(p,q): p,q are mixed strategies}.
- Want to define f(p,q) = (p',q') such that:
  - f is continuous. This means that changing p or q a little bit shouldn't cause p' or q' to change a lot.
  - Any fixed point of f is a Nash Equilibrium.
- Then Brouwer will imply existence of NE.

Try #1

- What about f(p,q) = (p',q') where p' is best response to q, and q' is best response to p?
- Problem: not necessarily well-defined:
  - E.g., penalty shot: if p = (0.5,0.5) then q' could be anything.

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Try #1

- What about f(p,q) = (p',q') where p' is best response to q, and q' is best response to p?
- Problem: also not continuous:
  - E.g., if p = (0.51, 0.49) then q' = (1,0). If p = (0.49,0.51) then q' = (0,1).

	Left	Right
Left	(0,0)	(1,-1)
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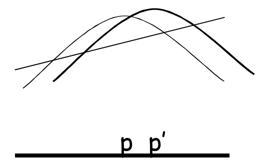
Instead we will use…

- f(p,q) = (p',q') such that:
  - q' maximizes [(expected gain wrt p) - $||q-q'||^2$]
  - p' maximizes [(expected gain wrt q) - $||p-p'||^2$]



Note: quadratic + linear = quadratic.

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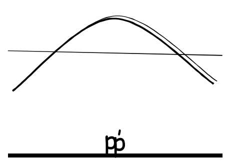

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  - q' maximizes [(expected gain wrt p) - $||q-q'||^2$]
  - p' maximizes [(expected gain wrt q) - $||p-p'||^2$]
- f is well-defined and continuous since quadratic has unique maximum and small change to p,q only moves this a little.
- Also fixed point = NE. (even if tiny incentive to move, will move little bit).
- So, that's it!

Internal regret and correlated equilibria

What if all players minimize regret?

- In zero-sum games, empirical frequencies quickly approaches minimax optimal.
- In general-sum games, does behavior quickly (or at all) approach a Nash equilibrium? (after all, a Nash Eq is exactly a set of distributions that are no-regret wrt each other).
- Well, unfortunately, no.
- (Even if it did, as we saw last time, you might not *want* to minimize regret in order to get other players to do what you want – e.g., ultimatum game)

A bad example for general-sum games

- Augmented Shapley game from [Z04]: "RPSF"
  - First 3 rows/cols are Shapley game (rock / paper / scissors but if both do same action then both lose).
  - 4th action "play foosball" has slight negative if other player is still doing r/p/s but positive if other player does 4th action too.
  - NR algs will cycle among first 3 and have no regret, but do worse than only Nash Equilibrium of both playing foosball.
- We didn't really expect this to work given how hard NE can be to find…

What *can* we say?

- If algorithms minimize "internal" or "swap" regret, then empirical distribution of play approaches *correlated* equilibrium.
  - Foster & Vohra, Hart & Mas-Colell,…
  - Though doesn't imply play is stabilizing.

What are internal regret and correlated equilibria?

More general forms of regret

1. "best expert" or "external" regret:
   - Given n strategies. Compete with best of them in hindsight.
2. "sleeping expert" or "regret with time-intervals":
   - Given n strategies, k properties. Let S_i be set of days satisfying property i (might overlap). Want to simultaneously achieve low regret over each S_i.
3. "internal" or "swap" regret: like (2), except that S_i = set of days in which we chose strategy i.

Internal/swap-regret

- E.g., each day we pick one stock to buy shares in.
  - Don't want to have regret of the form "every time I bought IBM, I should have bought Microsoft instead".
- Formally, regret is wrt optimal function $f:\{1,\ldots,N\}\rightarrow\{1,\ldots,N\}$ such that every time you played action j, it plays f(j).
- Motivation: connection to correlated equilibria.

Internal/swap-regret

"Correlated equilibrium"

- Distribution over entries in matrix, such that if a trusted party chooses one at random and tells you your part, you have no incentive to deviate.
- E.g., Shapley game.

	R	P	S
R	-1,-1	-1,1	1,-1
P	1,-1	-1,-1	-1,1
S	-1,1	1,-1	-1,-1

Internal/swap-regret

- If all parties run a low internal/swap regret algorithm, then empirical distribution of play is an apx correlated equilibrium.
  - Correlator chooses random time $t \in \{1,2,\ldots,T\}$. Tells each player to play the action j they played in time t (but does not reveal value of t).
  - Expected incentive to deviate: $\sum_j \Pr(j)(\text{Regret}|j)$ = swap-regret of algorithm
  - So, this says that correlated equilibria are a natural thing to see in multi-agent systems where individuals are optimizing for themselves

Internal/swap-regret, contd

Algorithms for achieving low regret of this form:

- Foster & Vohra, Hart & Mas-Colell, Fudenberg & Levine.
- Can also convert any "best expert" algorithm into one achieving low swap regret.
- Unfortunately, time to achieve low regret is linear in n rather than log(n)….

Internal/swap-regret, contd

Can convert any "best expert" algorithm A into one achieving low swap regret. Idea:

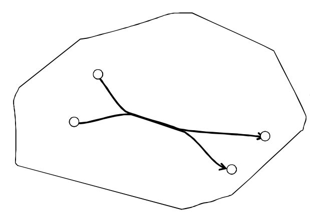
- Instantiate one copy A_i responsible for expected regret over times we play i.
- Each time step, if we play $p=(p_1,\ldots,p_n)$ and get cost vector $c=(c_1,\ldots,c_n)$, then A_i gets cost-vector $p_i c$.
- If each A_i proposed to play q_i, so all together we have matrix Q, then define p = pQ.
- Allows us to view p_i as prob we chose action i or prob we chose algorithm A_i.

Congestion games

- Many multi-agent interactions have structure. One nice class: Congestion Games
- Always have a pure-strategy equilibrium.
- Have a potential function s.t. whenever a player switches, potential drops by exactly that player's improvement.
  - So, best-response dynamics always gives an equilibrium.
- Let's start with an example.

Fair cost-sharing

Fair cost-sharing: n players in weighted directed graph G. Player i wants to get from s_i to t_i, and they share cost of edges they use with others.



Good equilibria, Bad equilibria

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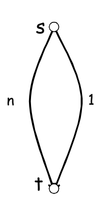

Good equilibrium: all use edge of cost 1. (cost 1/n per player)

Bad equilibrium: all use edge of cost n. (cost 1 per player)

Cost(bad equilib) = n·Cost(good equilib)

Good equilibria, Bad equilibria

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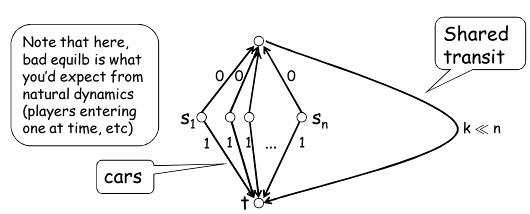


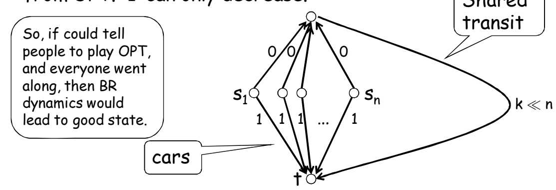
Price of Anarchy and Price of Stability

- Price of Anarchy: ratio of worst equilibrium to social optimum. (worst-case over games in class)
  - We saw for cost-sharing PoA = $\Omega(n)$. Also $O(n)$.
- Price of Stability: ratio of best equilibrium to social optimum. (worst-case over games in class)
  - For cost-sharing, PoS = $\Theta(\log n)$.
- Exact Potential function: Function Φ s.t. if player moves, potential changes by exactly as much as cost of player who moved.
  - Guarantees that best-response dynamics will reach Nash equilibrium

Potential functions and PoS

For cost-sharing, PoS = $O(\log n)$:

- Given state S, let n_e = # players on edge e. Cost(S) =
- Define potential $\Phi(S)$ =
- So, $\text{cost}(S) \leq \Phi(S) \leq \log(n) \times \text{cost}(S)$.
- Now consider best-response dynamics starting from OPT. Φ can only decrease.



Congestion games more generally

Game defined by n players and m resources.

- Each player i choses a set of resources (e.g., a path) from collection S_i of allowable sets of resources (e.g., paths from s_i to t_i).
- Cost of a resource j is a function $f_j(n_j)$ of the number n_j of players using it.
- Cost incurred by player i is the sum, over all resources being used, of the cost of the resource.
- Generic potential function:
- Best-response dynamics may take a long time to reach equil, but if gap between Φ and cost is small, can get to apx-equilib fast.