d/dt: A Tool for Reachability Analysis of Continuous and Hybrid Systems

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Outline

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2. Reachability problem and our approach
3. Reachability technique for non-linear continuous systems
4. Reachability technique for linear continuous systems
5. Analysis of hybrid systems: verification and controller synthesis
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Reachability Problem for Continuous Systems

The basic problem

$\dot{\mathbf{x}} = f(\mathbf{x})$; $\mathbf{x} \in \mathcal{X}$, bounded subset of R^n

$\mathbf{x}(0) \in F$, set of initial states

- Characterizing *the set of states reachable* from the *set F*
- Representation of reachable sets that can *be tested for intersection with other sets*

• Exact symbolic computation: applicable for restricted classes $\Rightarrow$ **Numerical approximation** of the reachable set

Reachability operators

For a given set $F \subseteq \mathcal{X}$ and a time interval $I = [t_1, t_2]$

$\delta_I(F)$: set of states reachable from F in time $t \in I$

$\delta_t(F)$: set of states reachable after exact amount of time t

$\delta(F) = \delta_{[0,\propto)}(F)$: reachable set

Semi-group property: $\delta_{[0,\, t_1+t_2]}(F) = \delta_{[0,\, t_2]}(\delta_{[0,\, t_1]}(F))$

Abstract Algorithm for Computing $\delta(F)$

$P^0 := F; \quad k := 0;$

repeat

  $k := k + 1;$

  $P^k := P^{k-1} \cup \delta_{[0,t]}(P^{k-1});$

until $(P^k = P^{k-1})$

Problems

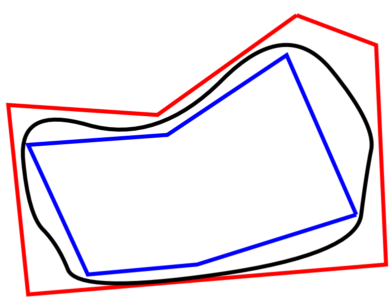
- Compute $\delta_{[0,t]}$ of a set
- Perform set *union*, *equivalence testing*

Approximation by Orthogonal Polyhedra

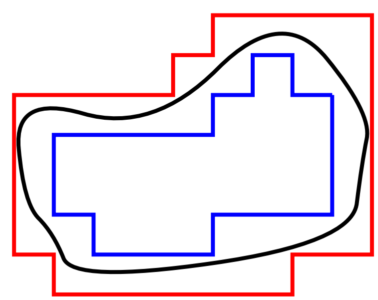
- Reachable sets are often *non-convex* $\Rightarrow$ difficulty in representing and manipulating arbitrary non-convex polyhedra

$\Rightarrow$ *Orthogonal Polyhedra*

- Canonical representation, relatively efficient manipulation
- Easy termination checking
- *Over*-approximation (*verification*), *under*-approximation (*synthesis*)



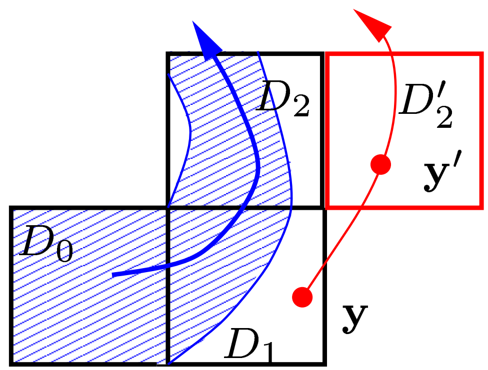
Arbitrary polyhedra



Orthogonal polyhedra

Approximation by Orthogonal Polyhedra (cont'd)

Accumulation of **over-approximation** *errors*

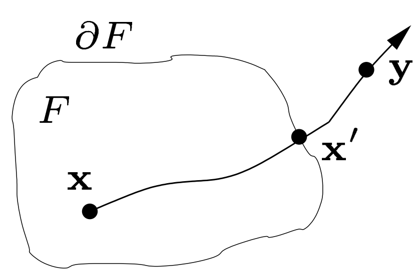


Reachability Technique for Non-Linear Continuous Systems

A non-linear continuous system: $\dot{\mathbf{x}} = f(\mathbf{x})$; F is the set of initial states

Face-lifting technique, inspired by [Greenstreet 96]

– Continuity of trajectories $\Rightarrow$ *computing from the boundary*



The set F is *polyhedral* $\Rightarrow$ boundary of F is the union of its *faces*

– Consider the evolution in the *outward normal direction of each face* of F

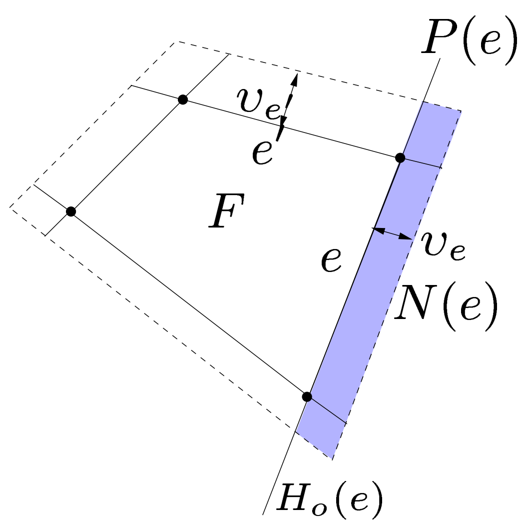
Face Lifting Technique

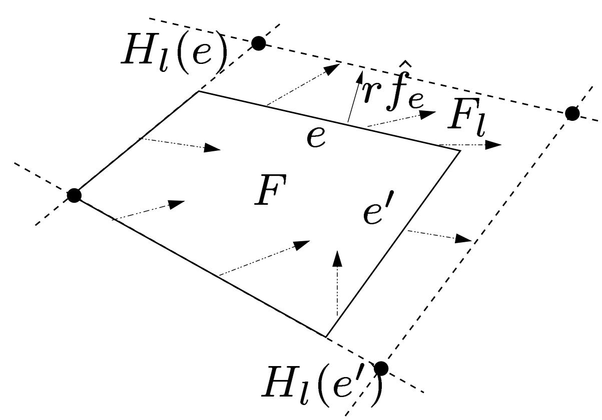
Step 1 - Rough approximation: neighborhood $N(F)$ such that all trajectories starting from F stay in $N(F)$ for at least r time.

Step 2 - More precise approximation

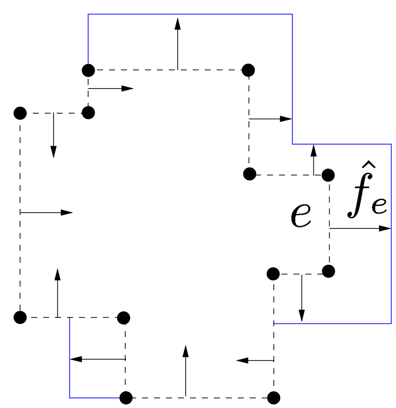
$f_e(\mathbf{x})$: projection of $f(\mathbf{x})$ on the outward normal vector $\mathbf{n}(e)$ of face e

$\hat{f}_e$: maximum of f_e over $N(e)$





Face Lifting using Orthogonal Polyhedra



Using orthogonal polyhedra:

- Faces can be systematically enumerated
- Orthogonal polyhedra are closed under the lifting operation

Reachability of Linear Continuous Systems

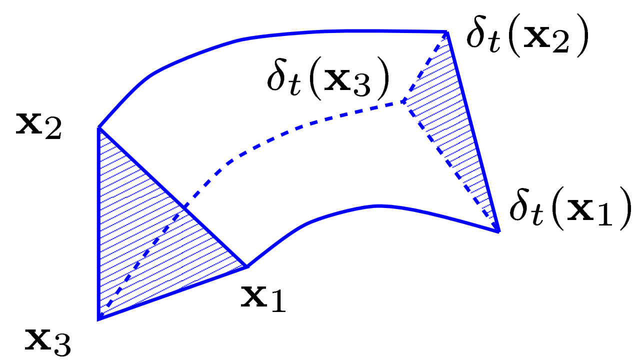
A linear continuous system: $\dot{\mathbf{x}} = A\mathbf{x}$, F is the set of initial states

$\delta_t(F) = e^{At}F$

Property: *Convexity is preserved*

$F = conv\{\mathbf{x}_1, \ \ldots \ , \mathbf{x}_m\}$; $\mathbf{x}_i$ are vertices of F

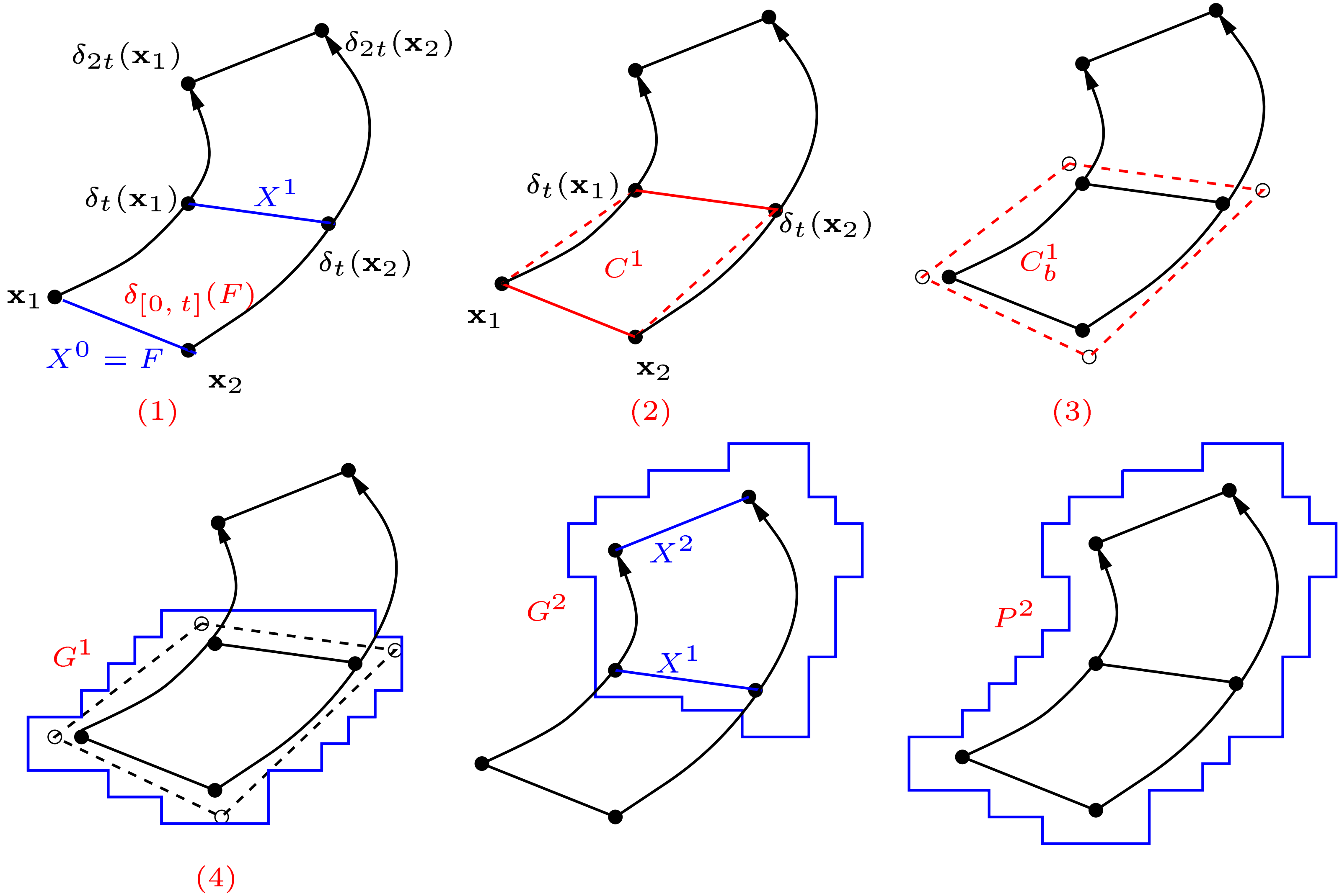
$\delta_t(F) = conv\{\delta_t(\mathbf{x}_1), \ \ldots \ , \delta_t(\mathbf{x}_m)\}$ with $\delta_t(\mathbf{x}_i) = e^{At}\mathbf{x}_i$



$\delta_t(F)$ can be computed by a finite number of integrations.

$\Rightarrow$ We exploit this property to approximate $\delta(F)$

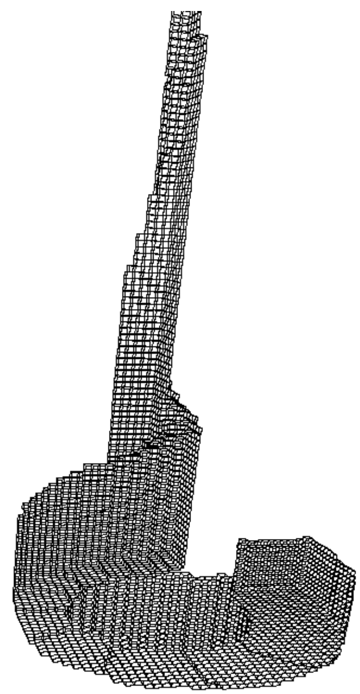
Approximate Computation of $\delta(F)$



Example

$\dot{\mathbf{x}} = A\mathbf{x}$, initial set $F = [0.025, 0.05] \times [0.1, 0.15] \times [0.05, 0.1]$

$$A = \begin{pmatrix} -1.0 & -4.0 & 0.0 \\ 4.0 & -1.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \end{pmatrix}$$



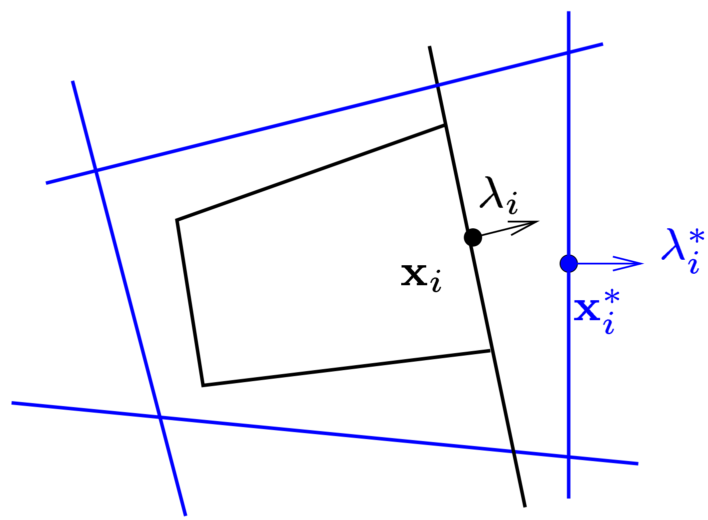
Linear Systems with Uncertain Input

$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{u}$, where $\mathbf{u} \in U$ (convex set)

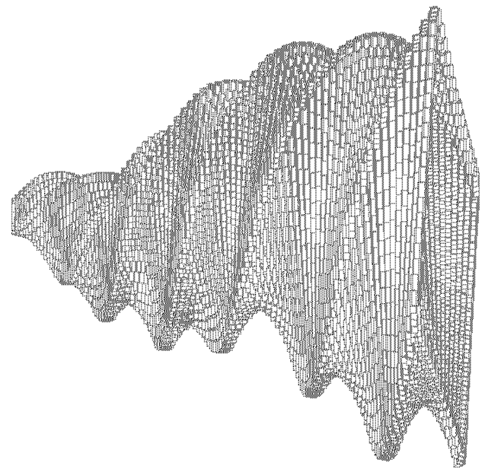
Approach: inspired by [Varaiya 98] using the Maximum Principle

Convex polyhedron $F = \bigcap\{\langle \lambda_i, \mathbf{x}\rangle \leq \gamma_i\}$ (*intersection of halfspaces*)

Face i: $\langle \lambda_i, \mathbf{x}\rangle = \gamma_i$ $\quad \mathbf{u}_i{}^* = argmax\{\ \langle \lambda_i, A\mathbf{x} + \mathbf{u}\rangle \mid \mathbf{u} \in U\ \}$



Example of Linear Continuous Systems with Input



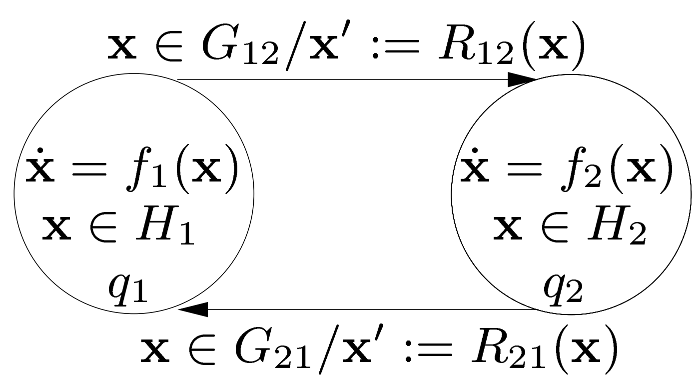
$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{u}, \quad \mathbf{u} \in U; \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{pmatrix}$$

Input set $U = [-0.5, 0.5] \times [-0.005, 0.005] \times [-0.5, 0.5] \times [-0.005, 0.005]$;

Initial set $F = [0, 2] \times [-1, 1] \times [0, 2] \times [-1, 1]$; [Kurzhanski and Valyi 97]

Hybrid Dynamical Systems

- several *modes* (*discrete states*, *locations*)
- continuous dynamics of the modes are defined by *differential equations*
- *staying* conditions (*'invariants'*) of each mode: convex polyhedra
- *switching* conditions (*'guards'*): convex polyhedra
- *reset functions* associated with transitions: affine of the form $R_{qq'}(\mathbf{x}) = D_{qq'}\mathbf{x} + J_{qq'}$



Reachability Analysis of Hybrid Systems

The state $(q, \mathbf{x})$ of the system can change in two ways:

- by **continuous dynamics**: location q remains constant, and $\mathbf{x}$ changes continuously according to the differential equation of location q
- by **discrete transitions**: location q changes, and $\mathbf{x}$ can be changed according to the reset function

Reachability procedure requires the computation of

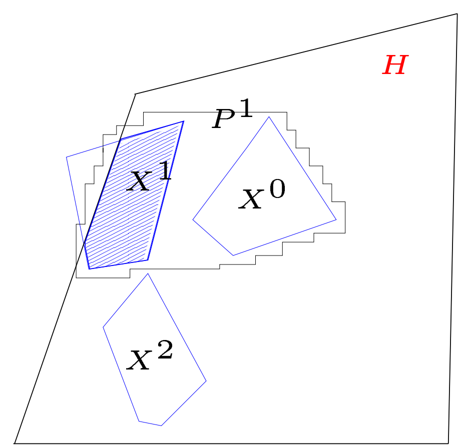
⋆ reachable sets by continuous dynamics (**continuous-successors**)

⋆ reachable sets by discrete transitions (**discrete-successors**)

Reachability Technique for Hybrid Systems

Computation of continuous-successors

- based on the computation of *reachable sets of continuous systems*
- takes into account the *staying condition* of the current location



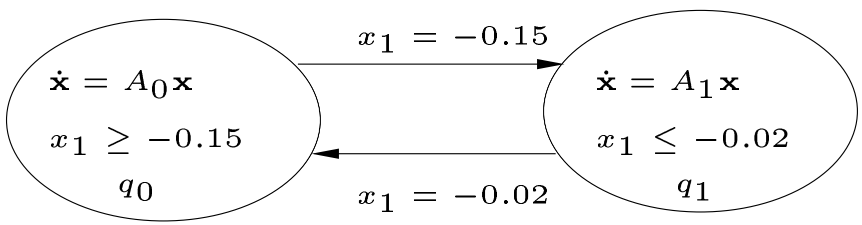
Computation of discrete-successors

- Set of discrete-successors of set F by transition from q to q':

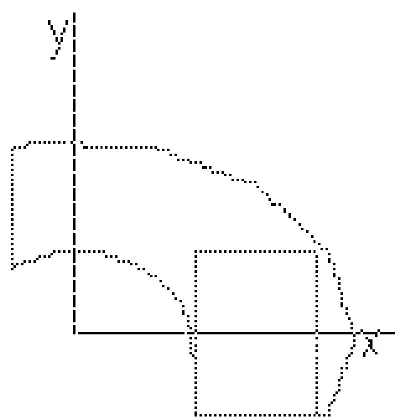
$$R_{qq'}(F \cap G_{qq'} \cap H_{q'})$$

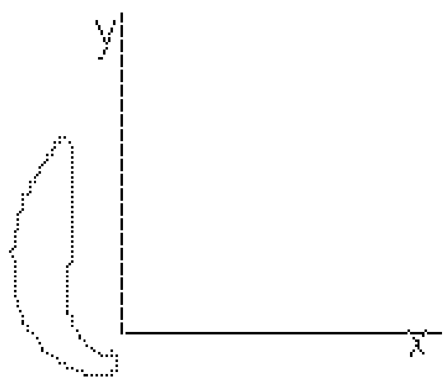
- *Boolean* and *geometric* operations on polyhedra

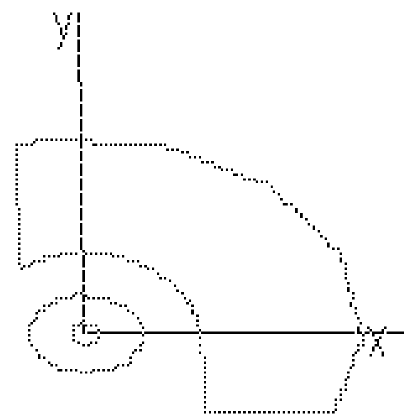
Example



$$A_0 = \begin{bmatrix} 0 & -0.6 \\ 3 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$$



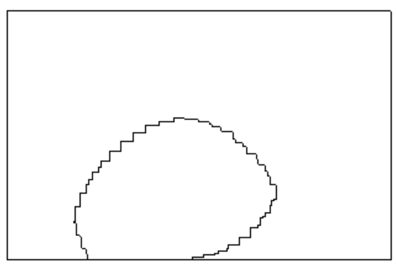




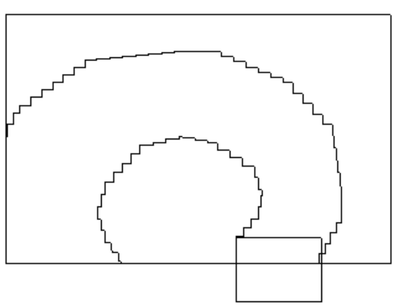
q_0 q_1 q_0

Example: $F\ Until\ G$

Given two subsets F and G. Calculate $F\ Until\ G$?



The states which can **stay in** F **forever**



The states which can **stay in** F and **reach** G

$A = \begin{pmatrix} -0.5 & 4.0 \\ -3.0 & -0.5 \end{pmatrix}$; $F = [-0.1, 0.1] \times [-0.03, 0.1]$;
$G = [0.02, 0.06] \times [-0.05, -0.02]$

The tool d/dt

Three functionalities

- **Reachability Analysis**
  - Linear continuous systems
  - Non-linear continuous systems
  - Hybrid systems
- **Safety verification of hybrid systems**
- **Safety switching controller synthesis** for hybrid systems with linear continuous dynamics

Future work

⋆ More efficient polyhedral approximation algorithms

⋆ Verification and controller synthesis for more general properties

Some Related Works

- Integrating Projection [Greenstreet 96]
- Ellipsoidal Techniques [Kurzhanski and Valyi 97; Kurzhanski and Varaiya 00]
- Flow-pipe Approximation [Chutinan and Krogh 99]
- Symbolic Method [Pappas, Lafferier and Yovine 99; Anai and Weipsfenning 01]
- Verification via mathematical programming [Bemporad and Morari 99]