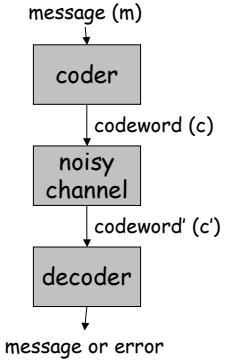
15-499:Algorithms and Applications

Error Correcting Codes I
- Overview
- Hamming Codes
- Linear Codes

General Model



Errors introduced by the noisy channel:
- changed fields in the codeword (e.g. a flipped bit)
- missing fields in the codeword (e.g. a lost byte). Called **erasures**

How the decoder deals with errors.
- **error detection** vs.
- **error correction**

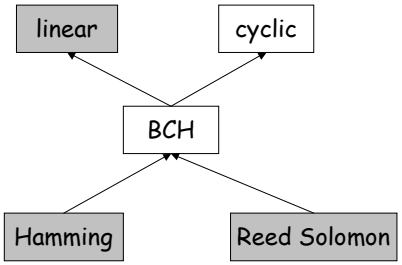
Applications

- **Storage**: CDs, DVDs, "hard drives",
- **Wireless**: Cell phones, wireless links
- **Satellite and Space**: TV, Mars rover, …
- **Digital Television**: DVD, MPEG2 layover
- **High Speed Modems**: ADSL, DSL, ..

Reed-Solomon codes are by far the most used in practice, including pretty much all the examples mentioned above.

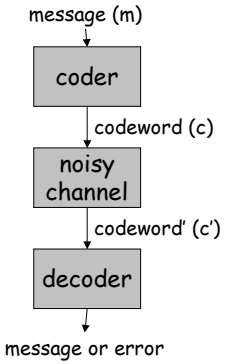
Algorithms for decoding are quite sophisticated.

Hierarchy of Codes



These are all "**block**" codes.

Block Codes



Each message and codeword is of fixed size

Σ = codeword alphabet

k = |m| **n** = |c| **q** = $|\Sigma|$

$C \subseteq \Sigma^n$ (codewords)

$\Delta(x,y)$ = number of positions s.t. $x_i \neq y_i$

$d = \min\{\Delta(x,y) : x,y \in C, x \neq y\}$

Code described as: $(n,k,d)_q$

Rate: k/n

Example of $(6,3,3)_2$ Block Code

messages	codewords
000	**000**000
001	**001**011
010	**010**101
011	**011**110
100	**100**110
101	**101**101
110	**110**011
111	**111**000

Σ = **{0,1}** (binary code)

k = |m| = **3**

n = |c| = **6**

q = $|\Sigma|$ = **2**

$C \subseteq \Sigma^n$ = {000000, 001011, …}

$d = \min\{\Delta(x,y) : x,y \in C, x \neq y\}$ = **?**

Rate: k/n = **.5**

Systematic Codes

message	codeword
000	**000**000
001	**001**011
010	**010**101
011	**011**110
100	**100**110
101	**101**101
110	**110**011
111	**111**000

Definition: A **Systematic code** is one in which the message appears in the codeword

Binary codes:

Today we will mostly be considering Σ = **{0,1}** and will sometimes use (n,k,d) as shorthand for $(n,k,d)_2$

In binary $\Delta(x,y)$ is often called the **Hamming distance**

Error Detection and Correction

s-symbol (bit) error detection: Can always detect errors of **any** s symbols (bits for binary codes).

Question: if d = 2, can you do 2-bit error detection? What about d = 3?

s-symbol (bit) error correction: Can always correct errors of **any** s symbols (bits for binary codes).

Question: if d = 2, can you do 1-bit error correction? What about d = 3?

Error Detection with a Parity Bit

A $(k+1,k,2)_2$ systematic code

Encoding:

$m_1m_2...m_k \Rightarrow m_1m_2...m_kp_{k+1}$

where $p_{k-1} = m_1 \oplus m_2 \oplus ... \oplus m_k$

d = 2 since the parity is always even (it takes two bit changes to go from one codeword to another).

Detects one-bit error since this gives odd parity

Cannot be used to correct 1-bit error since any odd-parity word is equal distance $\Delta = 1$ to k+1 valid codewords.

Error Correcting One Bit Messages

How many bits do we need to correct a one bit error on a one bit message?

We need 3 bits: a (n=3,k=1,d=3) code

Encode:

$m_1 \Rightarrow m_1m_1m_1$

Decode

The majority function (e.g. 101 $\Rightarrow$ 1)

d = 3 since 000 and 111 differ by 3.

In general:

We need $d \geq 3$ to correct one error.

Error Correcting Multibit Messages

We will first discuss **Hamming Codes**

Correct 1-bit errors, detect 2-bit errors.

Codes are of form: $(2^r-1, 2^r-1-r, 3)$ for any $r > 1$

e.g. (3,1,3), (7,4,3), (15,11,3), (31, 26, 3), …

which correspond to 2, 3, 4, 5, … "parity bits" (i.e. n-k)

The high-level idea is to "localize" the error.

Any specific ideas?

Hamming Codes: Encoding

Localizing error to top or bottom half 1xxx or 0xxx

m_{15}	m_{14}	m_{13}	m_{12}	m_{11}	m_{10}	m_9	p_8	m_7	m_6	m_5	m_4	m_3	m_2	m_1	p_0

$p_8 = m_{15} \oplus m_{14} \oplus m_{13} \oplus m_{12} \oplus m_{11} \oplus m_{10} \oplus m_9$

Localizing error to x1xx or x0xx

m_{15}	m_{14}	m_{13}	m_{12}	m_{11}	m_{10}	m_9	p_8	m_7	m_6	m_5	p_4	m_3	m_2	m_1	p_0

$p_4 = m_{15} \oplus m_{14} \oplus m_{13} \oplus m_{12} \oplus m_7 \oplus m_6 \oplus m_5$

Localizing error to xx1x or xx0x

m_{15}	m_{14}	m_{13}	m_{12}	m_{11}	m_{10}	m_9	p_8	m_7	m_6	m_5	p_4	m_3	p_2	m_1	p_0

$p_2 = m_{15} \oplus m_{14} \oplus m_{11} \oplus m_{10} \oplus m_7 \oplus m_6 \oplus m_3$

Localizing error to xxx1 or xxx0

m_{15}	m_{14}	m_{13}	m_{12}	m_{11}	m_{10}	m_9	p_8	m_7	m_6	m_5	p_4	m_3	p_2	p_1	p_0

$p_1 = m_{15} \oplus m_{13} \oplus m_{11} \oplus m_9 \oplus m_7 \oplus m_5 \oplus m_3$

binary repr. of position

Hamming Codes: Decoding

m_{15}	m_{14}	m_{13}	m_{12}	m_{11}	m_{10}	m_9	p_8	m_7	m_6	m_5	p_4	m_3	p_2	p_1	p_0

We don't need p_0, so we have a (15,11,?) code.

After transmission, we generate

$b_8 = p_8 \oplus m_{15} \oplus m_{14} \oplus m_{13} \oplus m_{12} \oplus m_{11} \oplus m_{10} \oplus m_9$

$b_4 = p_4 \oplus m_{15} \oplus m_{14} \oplus m_{13} \oplus m_{12} \oplus m_7 \oplus m_6 \oplus m_5$

$b_2 = p_2 \oplus m_{15} \oplus m_{14} \oplus m_{11} \oplus m_{10} \oplus m_7 \oplus m_6 \oplus m_3$

$b_1 = p_1 \oplus m_{15} \oplus m_{13} \oplus m_{11} \oplus m_9 \oplus m_7 \oplus m_5 \oplus m_3$

With no errors, these will all be zero

With one error $b_8b_4b_2b_1$ gives us the error location.

e.g. **0100** would tell us that $\mathbf{p_4}$ is wrong, and
1100 would tell us that $\mathbf{m_{12}}$ is wrong

Examples

m_7	m_6	m_5	p_4	m_3	p_2	p_1
1	1	0	1	0	1	1

m_7	m_6	m_5	p_4	m_3	p_2	p_1
0	1	1	0	0	0	1

Hamming Codes

Can be generalized to any power of 2

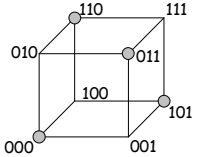
- $n = 2^r - 1$ (15 in the example)
- $(n-k) = r$ (4 in the example)
- $d = 3$ (discuss later)
- Gives $(2^r-1, 2^r-1-r, 3)$ code

Extended Hamming code

- Add back the parity bit at the end (p_0)
- Gives $(2^r, 2^r-1-r, 4)$ code
- Can correct one error and detect 3.

Lower bound on parity bits

Consider codewords as vertices on a hypercube.



codeword

d = 2 = min distance
n = 3 = dimensionality
2^n = 8 = number of nodes

The distance between nodes on the hypercube is the Hamming distance Δ. The minimum distance is d.

001 is equidistance from 000, 011 and 101.

For s-bit error detection $d \geq s + 1$

For s-bit error correction $d \geq 2s + 1$

Lower bound on parity bits

How many nodes in hypercube do we need so that d = 3?

Each of the 2^k codewords eliminates n neighbors plus itself, i.e. n+1

$$\begin{aligned} 2^n &\geq (n+1)2^k \\ n &\geq k + \log_2(n+1) \\ n &\geq k + \lceil \log_2(n+1) \rceil \end{aligned}$$

In previous hamming code $15 \geq 11 + \lceil \log_2(15+1) \rceil = 15$

Hamming Codes are called **perfect codes** since they match the lower bound exactly

Lower bound on parity bits

What about fixing 2 errors (i.e. d=5)?

Each of the 2^k codewords eliminates itself, its neighbors and its neighbors' neighbors, giving: $1+\binom{n}{1}+\binom{n}{2}$

$$\begin{aligned} 2^n &\geq (1+n+n(n-1)/2)2^k \\ n &\geq k + \log_2(1+n+n(n-1)/2) \\ &\geq k + 2\log_2 n - 1 \end{aligned}$$

Generally to correct s errors:

$$n \geq k + \log_2(1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{s})$$

Lower Bounds: a side note

The lower bounds assume random placement of bit errors.

In practice errors are likely to be less than random, e.g. evenly spaced or clustered:

Can we do better if we assume **regular errors**?

We will come back to this later when we talk about **Reed-Solomon** codes.

Linear Codes

If Σ is a field, then Σ^n is a **vector space**

Definition: *C* is a linear code if it is a linear subspace of Σ^n of dimension k.

- For example for n = 3, and k = 2, we have a plane cutting through a cube.

This means that there is a set of k basis vectors $v_i \in \Sigma^n$ $(1 \leq i \leq k)$ that span the subspace.

i.e. every codeword can be written as:

$c = a_1 v_1 + \ldots + a_k v_k \quad a_i \in \Sigma$

Linear Codes

Basis vectors for the $(7,4,3)_2$ Hamming code:

		m_7	m_6	m_5	p_4	m_3	p_2	p_1	
v_1	=	1	0	0	1	0	1	1	
v_2	=	0	1	0	1	0	1	0	
v_3	=	0	0	1	1	0	0	1	
v_4	=	0	0	0	0	1	1	1	
		——			——				p_4
		——					——		p_2
		—	—		—	—			p_1

Linear Codes

Basis vectors for the $(7,4,3)_2$ Hamming code:

		m_7	m_6	m_5	p_4	m_3	p_2	p_1
v_1	=	1	0	0	1	0	1	1
v_2	=	0	1	0	1	0	1	0
v_3	=	0	0	1	1	0	0	1
v_4	=	0	0	0	0	1	1	1

How can we see that d = 3?

For all binary linear codes, the minimum distance is equal to the least weight non-zero codeword.

Consider a codeword c_1, then $c_2 + c_1 = c_3$, and $c_3 - c_2 = c_1$ so $\Delta(c_3,c_2) = \text{weight}(c_1)$

Generator and Parity Check Matrices

Generator Matrix:

A k x n matrix **G** such that: $C = \{xG, x \in \sum^k\}$

Made from stacking the basis vectors

Parity Check Matrix:

A (n – k) x n matrix **H** such that: $C = \{y \in \sum^n \mid Hy^T = 0\}$

Codewords are the nullspace of H

These **always exist for linear codes**

Theorem: H and G generate same code iff $HG^T = 0$

Proof: $0 = Hy^T = H(xG)^T = H(G^Tx^T) = (HG^T)x^T$
only true for all x if $HG^T = 0$

Advantages of Linear Codes

- Encoding is efficient (vector-matrix multiply)
- Error detection is efficient (vector-matrix multiply)
- **Syndrome** (Hy^T) has error information
- Gives q^{n-k} sized table for decoding
  Useful if n-k is small

Example and "Standard Form"

For the Hamming (7,4,3) code:

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

By swapping columns 4 and 5 it is in the form I_k, A.
A code with a matrix in this form is **systematic**, and
G is in "**standard form**"

$$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

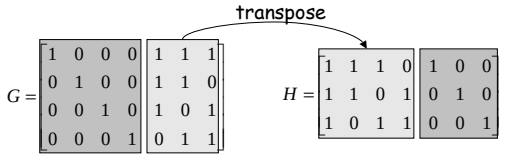
Relationship of G and H

If G is in standard form $[I_k, A]$
then $H = [A^T, I_{n-k}]$

Proof:

$HG^T = A^T I_k + I_{n-k} A^T = A^T + A^T = 0$

Example of (7,4,3) Hamming code:



The d of linear codes

Theorem: Linear codes have distance d if every set of (d-1) columns of H are linearly independent, but there is a set of d columns that are linearly dependent.

Proof summary: if d columns are linearly dependent then there exist two codewords that differ in the d bits corresponding to those columns that make the same contribution to the syndrome.

Dual Codes

For every code with

$G = I_k, A$ and $H = A^T, I_{n-k}$

we have a **dual code** with

$G = I_{n-k}, A^T$ and $H = A, I_k$

The dual of the Hamming codes are the **binary simplex codes**: $(2^r-1, r, 2^{r-1}-r)$

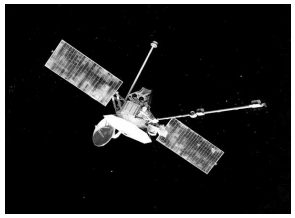
The dual of the extended Hamming codes are the **first-order Reed-Muller** codes: $(2^r, r+1, 2^{r-1})$,

Note that these codes are **highly redundant** and can fix many errors.

NASA Mariner:

Deep space probes from 1969-1977.

Mariner 10 shown



Used (32,6,16) Reed Muller code (r = 5)

Rate = 6/32 = .1875 (only 1 out of 5 bits are useful)

Can fix up to 7 bit errors per 32-bit word

How to find the error locations

Hy^T is called the **syndrome** (no error if 0).

In **general** we can find the error location by creating a table that maps each syndrome to a set of error locations.

Theorem: assuming the number of errors $s \leq 2d-1$ every syndrome value corresponds to a unique set of error locations.

Proof: Exercise.

Table has q^{n-k} entries, each of size at most n bits (i.e. keep a bit vector of locations).