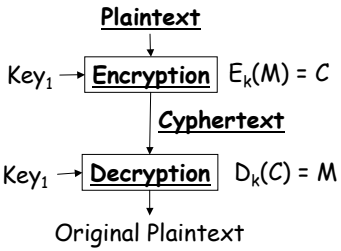
15-499:Algorithms and Applications

Cryptography III
- Private Key Cryptosystems
- Public Key Cryptosystems

Cryptography Outline

Introduction: terminology and background
Primitives: one-way hash functions, trapdoors, …
Protocols: digital signatures, key exchange, ..
Number Theory: groups, fields, …
➡ **Private-Key Algorithms:** Rijndael, DES, RC4
Cryptanalysis: Differential, Linear
Public-Key Algorithms: Knapsack, RSA, El-Gamal, Blum-Goldwasser
Case Studies: Kerberos, Digital Cash

Private Key Algorithms



What granularity of the message does E_k encrypt?

Private Key Algorithms

Block Ciphers: blocks of bits at a time
- DES (Data Encryption Standard)
  Banks, linux passwords (almost), SSL, kerberos, …
- Blowfish (SSL as option)
- IDEA (used in PGP, SSL as option)
- Rijdael (AES) – **the new standard**

Stream Ciphers: one bit (or a few bits) at a time
- RC4 (SSL as option)
- PKZip
- Sober, Leviathan, Panama, …

Private Key: Block Ciphers

Encrypt one block at a time (e.g. 64 bits)

$c_i = f(k,m_i)$ $m_i = f'(k,c_i)$

Keys and blocks are often about the same size.

Equal message blocks will encrypt to equal codeblocks

- Why is this a problem?

Various ways to avoid this:

- E.g. $c_i = f(k,c_{i-1} \oplus m_i)$
  "**Cipher block chaining**" (CBC)

Why could this still be a problem?

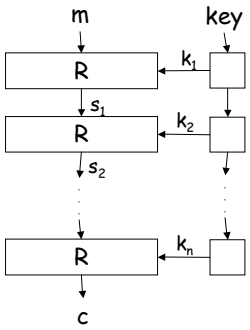
Solution: attach random block to the front of the message

Security of block ciphers

Ideal:

- k-bit -> k-bit key-dependent subsitution (i.e. "random permutation")
- If keys and blocks are k-bits, can be implemented with 2^{2k} entry table.

Iterated Block Ciphers



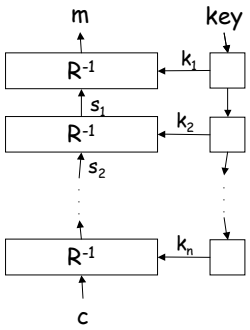
Consists of n **rounds**

R = the "round" function

s_i = state after round i

k_i = the i^{th} round key

Iterated Block Ciphers: Decryption

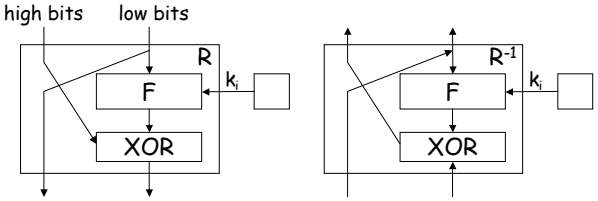


Run the rounds in reverse.

Requires that R has an inverse.

Feistel Networks

If function is not invertible rounds can still be made invertible. Requires 2 rounds to mix all bits.



Used by DES (the Data Encryption Standard)

Product Ciphers

Each round has two components:

- **Substitution** on smaller blocks
  Decorrelate input and output: "confusion"
- **Permutation** across the smaller blocks
  Mix the bits: "diffusion"

Substitution-Permutation Product Cipher

Avalanch Effect: 1 bit of input should affect all output bits, ideally evenly, and for all settings of other in bits

Rijndael

Selected by AES (Advanced Encryption Standard, part of NIST) as the new private-key encryption standard.

Based on an open "competition".

- Competition started Sept. 1997.
- Narrowed to 5 Sept. 1999
  - MARS by IBM, RC6 by RSA, Twofish by Counterplane, Serpent, and Rijdael
- Rijndael selected Oct. 2000.
- Official Oct. 2001? (AES page on Rijdael)

Designed by Rijmen and Daemen (Dutch)

Goals of Rijndael

Resistance against known attacks:

- Differential cryptanalysis
- Linear cryptanalysis
- Truncated differentials
- Square attacks
- Interpolation attacks
- Weak and related keys

Speed + Memory efficiency across platforms

- 32-bit processors
- 8-bit processors (e.g smart cards)
- Dedicated hardware

Design simplicity and clearly stated security goals

High-level overview

An iterated block cipher with
- 10–14 rounds,
- 128-256 bit blocks, and
- 128-256 bit keys

Mathematically reasonably sophisticated

Blocks and Keys

The blocks and keys are organized as matrices of bytes. For the 128-bit case, it is a 4x4 matrix.

$$\begin{pmatrix} b_0 & b_4 & b_8 & b_{12} \\ b_1 & b_5 & b_9 & b_{13} \\ b_2 & b_6 & b_{10} & b_{14} \\ b_3 & b_7 & b_{11} & b_{15} \end{pmatrix} \qquad \begin{pmatrix} k_0 & k_4 & k_8 & k_{12} \\ k_1 & k_5 & k_9 & k_{13} \\ k_2 & k_6 & k_{10} & k_{14} \\ k_3 & k_7 & k_{11} & k_{15} \end{pmatrix}$$

Data block Key

$b_0, b_1, \ldots, b_{15}$ is the order of the bytes in the stream.

Galois Fields in Rijdael

Uses GF(2^8) over bytes.

The irreducible polynomial is:

$M(x) = x^8 + x^4 + x^3 + x + 1$ or 100011011 or 0x11B

Also uses degree 3 polynomials with coefficients from GF(2^8).

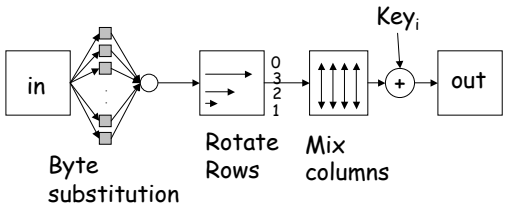
These are kept as 4 bytes (used for the columns)

The polynomial used as a modulus is:

$M(x) = 00000001x^4 + 00000001$ or $x^4 + 1$

Not irreducible, but we only need to find inverses of polynomials that are relatively prime to it.

Each round



The inverse runs the steps and rounds backwards. Each step must be reversible!

Byte Substitution

Non linear: $y = b^{-1}$ (done over GF(2^8))

Linear: $z = Ay + b$ (done over GF(2), *i.e.*, binary)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ & & & \vdots & & & & \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

To invert the substitution:

$y = A^{-1}(z - b)$ (the matrix A is nonsingular)

$b = y^{-1}$ (over GF(2^8))

Mix Columns

For each column:

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = C(x) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{pmatrix}$$

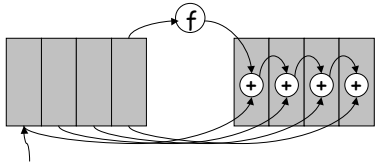
This corresponds to 4 polynomial multiplies each of degree 4 using GF(2^8) as the coefficients, and then reduced by M(x) = (x^4+1)

For example:

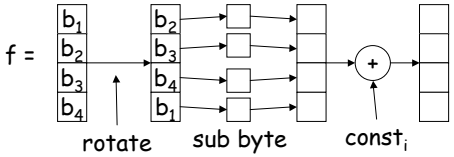
a′ = (ax^3+bx^2+cx+d)($2x^3+3x^2+x+1$) mod (x^4+1)

M(x) is not irreducible, but the rows of C and M(x) are coprime, so the transform can be inverted.

Generating the round keys

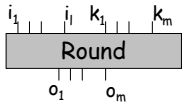


Words corresponding to columns of the key



Linear Cryptanalysis

A known plaintext attack used to extract the key



Consider a linear equality involving i, o, and k

- e.g.: $k_1 \oplus k_6 = i_2 \oplus i_4 \oplus i_5 \oplus o_4$

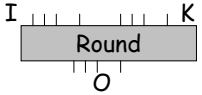
To be secure this should be true with p = .5 (probability over all inputs and keys)

If true with p = 1, then linear and easy to break

If true with p = .5 + ε then you might be able to use this to help break the system

Differential Cryptanalysis

A chosen plaintext attack used to extract the key



Considers fixed "differences" between inputs, $\Delta_I = I_1 - I_2$, and sees how they propagate into differences in the outputs, $\Delta_O = O_1 - O_2$. "difference" is often exclusive OR

Assigns probabilities to different keys based on these differences. With enough and appropriate samples (I_1, I_2, O_1, O_2), the probability of a particular key will converge to 1.

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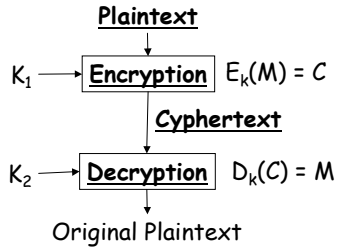
Case Studies: Kerberos, Digital Cash

Public Key Cryptosystems

Introduced by Diffie and Hellman in 1976.



Public Key systems
K_1 = public key
K_2 = private key

Digital signatures
K_1 = private key
K_2 = public key

Typically used as part of a more complicated protocol.

One-way trapdoor functions

Both Public-Key and Digital signatures make use of one-way trapdoor functions.

Public Key:
- Encode: c = f(m)
- Decode: m = f^{-1}(c) using trapdoor

Digital Signatures:
- Sign: c = f^{-1}(m) using trapdoor
- Verify: m = f(c)

Example of SSL

SSL (**Secure Socket Layer**) is the standard for the web (**https**).

Protocol (somewhat simplified):

B->A: hello www.amazon.com I would like a secure connection

A->B: Hi, I'm amazon.com, $|\text{amazon.com}|_{\text{verisign}}$

B->A: Prove it

A->B: Bob this is amazon, $\langle[\text{Bob this is amazon}]\rangle_{\text{amazon's private key}}$

B->A: ok Amazon, here is a shared-key, $\{\text{key}\}_{\text{amazon's public key}}$

A->B: $(\text{message},[\text{message}])_{\text{key}}$

$|h|_{\text{issuer}}$ = Certificate
= Issuer, <h,h's public key, time stamp>$_{\text{issuer's private key}}$

$\langle\ldots\rangle_{\text{private key}}$ = Digital signature $\{\ldots\}_{\text{public key}}$ = Public encryption
[..] = Secure Hash $(\ldots)_{\text{key}}$ = Private encryption

Public Key History

Some algorithms

- **Merkle-Hellman**, 1978, based on "knapsack problem"
- **McEliece**, 1978, based on algebraic coding theory
- **RSA**, 1978, based on factoring
- **Rabin**, 1979, security can be reduced to factoring
- **ElGamal**, 1985, based on Discrete logs
- **Blum-Goldwasser,** 1985, based on quadratic residues
- **Elliptic curves**, 1985, discrete logs over Elliptic curves
- **Chor-Rivest,** 1988, based on knapsack problem
- **NTRU**, 1996, based on Lattices
- **XTR,** 2000, based on discrete logs of a particular field

Diffie-Hellman Key Exchange

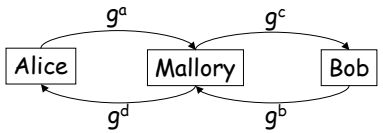
A group (G,*) and a primitive element (generator) g is made public.

- Alice picks a, and sends g^a to Bob
- Bob picks b and sends g^b to Alice
- The shared key is g^{ab}

Note this is easy for Alice or Bob to compute, but assuming discrete logs are hard is hard for anyone else to compute.

Can someone see a problem with this protocol?

Person-in-the-middle attack



$\text{Key}_1 = g^{ad}$ $\text{Key}_1 = g^{cb}$

Mallory gets to listen to everything.

Merkle-Hellman

Gets "security" from the **Subet Sum** (also called **knapsack**) which is NP-hard to solve in general.

Subset Sum (Knapsack): Given a sequence W = {w_0,w_1, …,w_{n-1}}, $w_i \in Z$ of weights and a sum S, calculate a boolean vector B, such that:

$$\sum_{i=0}^{i<n} B_i W_i = S$$

Even deciding if there is a solution is NP-hard.

Merkle-Hellman

W is **superincreasing** if: $w_i \geq \sum_{j=0}^{j-1} w_j$

It is easy to solve the subset-sum problem for superincreasing W in O(n) time.

Main idea:

- Hide the easy case by multiplying each w_i by a constant **a** modulo a prime **p**

$$w_i' = a * w_i \bmod p$$

- **Public key**: W' $\quad E_{W'}(M) = \sum_{i=0}^{i<n} w_i' m_i$
- **Private key**: a, p : $D_w(S)$ = solve for B_i

Merkle Hellman: Problem

Was broken by Shamir in 1984.

Shamir showed how to use integer programming to solve the particular class of Subset Sum problems in polynomial time.

Lesson: don't leave your trapdoor loose.

RSA

Invented by Rivest, Shamir and Adleman in 1978

Based on **difficulty of factoring**.

Used to **hide** the size of a group Z_n^* since:

. $$\left|Z_n^*\right| = \phi(n) = n \prod_{p|n} (1 - 1/p)$$

Factoring has not been reduced to RSA

- an algorithm that generates m from c does not give an efficient algorithm for factoring

On the other hand, factoring has been reduced to finding the private-key.

- there is an efficient algorithm for factoring given one that can find the private key.

RSA Public-key Cryptosystem

What we need:

- p and q, primes of approximately the same size
- n = pq
  $\phi(n) = (p-1)(q-1)$
- $e \in Z_{\phi(n)}^*$
- $d = e^{-1} \bmod \phi(n)$

Public Key: (e,n)
Private Key: d

Encode:
$m \in Z_n$
$E(m) = m^e \bmod n$

Decode:
$D(c) = c^d \bmod n$

RSA continued

Why it works:

$D(c) = c^d \bmod n$
$= m^{ed} \bmod n$
$= m^{1 + k(p-1)(q-1)} \bmod n$
$= m^{1 + k\phi(n)} \bmod n$
$= m(m^{\phi(n)})^k \bmod n$
$= m$

Why is this argument not quite sound?
What if $m \notin Z_n^*$ then $m^{\phi(n)} \neq 1 \bmod n$
Answer 1: Not hard to show that it still works.
Answer 2: jackpot – you've factored n

RSA computations

To **generate the keys**, we need to

- **Find two primes p and q**. Generate candidates and use primality testing to filter them.
- **Find e^{-1} mod (p-1)(q-1)**. Use Euclid's algorithm. Takes time $\log^2(n)$

To **encode and decode**

- **Take m^e or c^d**. Use the power method. Takes time log(e) $\log^2(n)$ and log(d) $\log^2(n)$.

In practice e is selected to be small so that encoding is fast.

Security of RSA

Warning:

- Do not use this or any other algorithm naively!

Possible security holes:

- Need to use "safe" primes p and q. In particular p-1 and q-1 should have large prime factors p_1 and q_1, and p_1 and q_1 should have large factors, …
- p and q should not have the same number of digits. Can use a middle attack starting at sqrt(n).
- E cannot be too small
- Don't use same n for different e's.
- You should always "pad"

Algorithm to factor given d and e

If an attacker has an algorithm that generates d from e, then he/she can factor n in PPT. Variant of the Rabin-Miller primality test.

Function TryFactor(e,d,n)

1. write **ed** – 1 as $2^s r$, r odd
2. choose w at random < n
3. $v = w^r \bmod n$
4. if v = 1 then return(**fail**)
5. while $v \neq 1 \bmod n$
6. $v_0 = v$
7. $v = v^2 \bmod n$
8. if $v_0 = n - 1$ then return(**fail**)
9. return(**pass,** $\gcd(v_0 + 1, n)$)

LasVegas algorithm
Probability of pass is > .5.
Will return p or q if it passes.
Try until you pass.

RSA Performance

Performance: (600Mhz PIII) (from: ssh toolkit):

Algorithm	Bits/key		Mbits/sec
RSA Keygen	1024	.35sec/key	
	2048	2.83sec/key	
RSA Encrypt	1024	1786/sec	3.5
	2048	672/sec	1.2
RSA Decrypt	1024	74/sec	.074
	2048	12/sec	.024
ElGamal Enc.	1024	31/sec	.031
ElGamal Dec.	1024	61/sec	.061
DES-cbc	56		95
twofish-cbc	128		140
Rijdael	128		180

RSA in the "Real World"

Part of many standards: PKCS, ITU X.509, ANSI X9.31, IEEE P1363

Used by: SSL, PEM, PGP, Entrust, …

The standards specify many details on the implementation, e.g.

- e should be selected to be small, but not too small
- "multi prime" versions make use of n = pqr… this makes it cheaper to decode especially in parallel (uses Chinese remainder theorem).

Factoring in the Real World

Quadratic Sieve (QS):

$$T(n) = e^{(1+o(n))(\ln n)^{1/2}(\ln(\ln n))^{1/2}}$$

- Usedin 1994 to factor a 129 digint (428-bit) number. 1600 Machines, 8 months.

Number field Sieve (NFS):

$$T(n) = e^{(1.923+o(1))(\ln n)^{1/3}(\ln(\ln n))^{2/3}}$$

- Used in 1999 to factor 155 digit (512-bit) number. 35 CPU years. At least 4x faster than QS

ElGamal

Based on the difficulty of the discrete log problem.
Invented in 1985
Digital signature and Key-exchange variants
- Digital signature is AES standard
- Public Key used by TRW (avoided RSA patent)

Works over various groups
- Z_p,
- Multiplicative group GF(p^n),
- Elliptic Curves

ElGamal Public-key Cryptosystem

(G,*) is a group
- α a generator for G
- a $\in Z_{|G|}$
- $\beta = \alpha^a$

G is selected so that it is hard to solve the discrete log problem.

Public Key: (α, β) and some description of G
Private Key: a

Encode:
Pick random k $\in Z_{|G|}$
E(m) = (y_1, y_2)
= $(\alpha^k, m * \beta^k)$

Decode:
D(y) = $y_2 * (y_1^a)^{-1}$
= $(m * \beta^k) * (\alpha^{ka})^{-1}$
= $m * \beta^k * (\beta^k)^{-1}$
= m

You need to know a to easily decode y!

ElGamal Example

N = 587
G = 5

Probabilistic Encryption

For RSA one message goes to one cipher word. This means we might gain information by running $E_{public}(M)$.

Probabilistic encryption maps every M to many C randomly. Cryptanalysist cant tell whether $C = E_{public}(M)$.

ElGamal is an example (based on the random k), but it doubles the size of message.

BBS "secure" random bits

BBS (Blum, Blum and Shub, 1984)

- Based on difficulty of factoring, or finding square roots modulo n = pq.

Fixed

- p and q are primes such that p = q = 3 (mod 4)
- n = pq (is called a Blum integer)

For a particular bit seq.

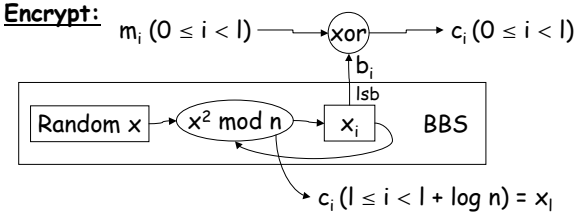
- **Seed:** random x relatively prime to n.
- **Initial state:** $x_0 = x^2$
- **i^{th} state:** $x_i = (x_{i-1})^2$
- **i^{th} bit:** lsb of x_i

Note that: $x_0 = x_i^{-2^i \bmod \phi(n)} (\bmod n)$
Therefore knowing p and q allows us to find x_0 from x_i

Blum-Goldwasser: A stream cypher

Public key: n (= pq) **Private key**: p or q

Encrypt:



Decrypt:

Using p and q, find $x_0 = x_i^{-2^i \bmod (p-1)(q-1)} (\bmod n)$

Use this to regenerate the $\mathbf{b_i}$ and hence $\mathbf{m_i}$

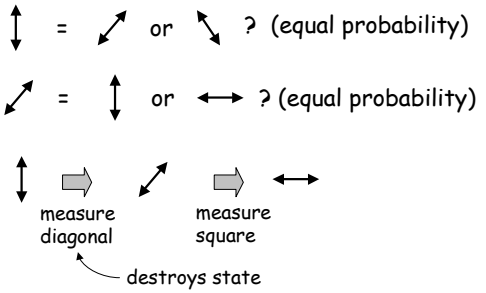
Quantum Cryptography

In quantum mechanics, there is no way to take a measurement without potentially changing the state. E.g.

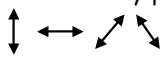
- Measuring position, spreads out the momentum
- Measuring spin horizontally, "spreads out" the spin probability vertically

Related to Heisenberg's uncertainty principal

Using photon polarization



Quantum Key Exchange

1. Alice sends bob photon stream randomly polarized in one of 4 polarizations: 
2. Bob measures photons in random orientations
   e.g.: **X + + X X X + X** (orientations used)
   \ | - \ / / - \ (measured polarizations)
   and tells Alice in the open what orientations he used, but not what he measured.
3. Alice tells Bob in the open which are correct
4. Bob and Alice keep the correct values

Susceptible to a **man-in-the-middle** attack

In the "real world"

Not yet used in practice, but experiments have verified that it works.

IBM has working system over 30cm at 10bits/sec.

More recently, up to 10km of fiber.