15-853:Algorithms in the Real World

Suffix Trees

Exact string searching

Given a text T of length n and pattern P of length m

"Quickly" find an occurrence (or all occurences) of P in T

A Naïve solution:

Compare P with T[i…i+m] for all i — O(nm) time

How about O(n+m) time? (Knuth Morris Pratt)

How about O(n) preprocessing time and O(m) search time?

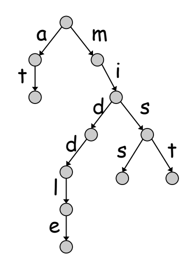
Notation:

Capital letters for strings: A, B, S, T, …

Lower-case letters for characters: a, b, c, …

TRIEs

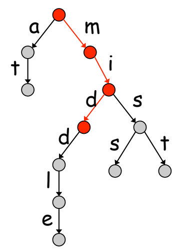
Dictionary = {at, middle, miss, mist}



TRIEs (searching)

Consider searching a string of length m to see if it is a **prefix** of an element in the dictionary

Search time depends on implementation of nodes



e.g., Search("mid")

TRIEs (node implementation)

Consider searching a string of length m to see if it is a **prefix** of an element in the dictionary

Consider an n-node trie over alphabet Σ, with $|\Sigma|=k$.

Implementation choices:

- Array per node: $O(nk)$ space, $O(m)$ time search
- Tree per node: $O(n)$ space, $O(m \log k)$ time search
- Hash children: $O(n)$ space, $O(m)$ time

  Don't need a separate table per node — can hash node pointer and child character

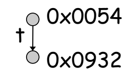


Table.Lookup((0x0054,t)) = 0x0932

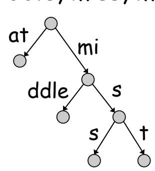
Compressed Tries

Also called PATRICIA tries/trees or radix trees.

All nodes with a single child are collapsed

Edges now represent substrings

Dictionary = {at, middle, miss, mist}

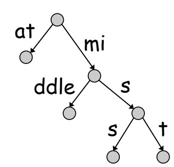


Takes less space in practice

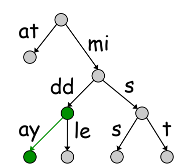
Insertion

Inserting string S into a compressed trie

- Find longest common prefix
- Split edge if necessary
- Add remaining suffix



Insert("midday")



Takes $O(|S|)$ time

Using Suffixes

Tries allow prefix searching. What about searching for any substring?

Store all the suffixes in a (compressed) trie

S = "nananabanana"

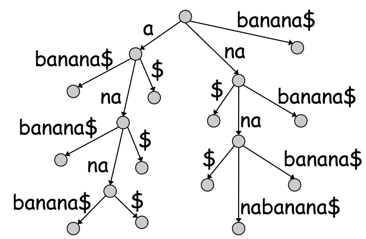
⇒ Dictionary = {nananabanana, ananabanana, nanabanana, anabanana, nabanana, abanana, banana, anana, nana, ana, na, a}

Typically use special character ($) at the end of a string to make sure every entry has a leaf.

Suffix Trees

A compressed trie on all suffixes of a string

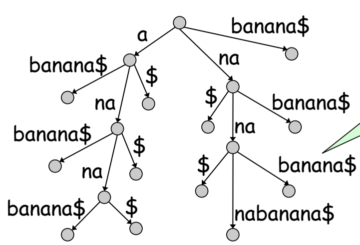
S = "nananabanana$"



Suffix Tree Representation

How do we store a suffix tree in O(n) space?

S = "nananabanana$" |S| = n



"banana$"=S[7..13]
store pointers 7,13

Suffix Tree Construction

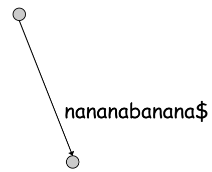
Simple algorithm:

T = empty
for i = 1 to n
   insert(S[i..n],T)

Takes $O(n^2)$ time in the worst case

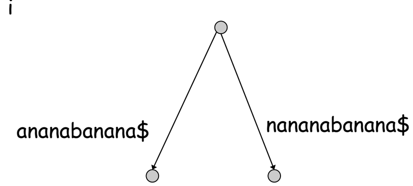
Suffix Tree Construction

S = nananabanana$
↑
i



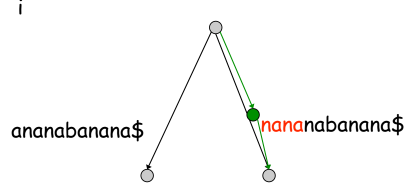
Suffix Tree Construction

S = nananabanana$
↑
i



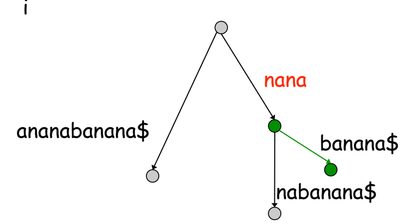
Suffix Tree Construction

S = nananabanana$
↑
i



Suffix Tree Construction

S = nananabanana$
↑
i



inserting "nanabanana$" matches "nana": 4 comparisons
inserting "anabanana$" will match "ana": 3 comparisons
then nabanana$ and abanana$...

First Improvement

Consider insert of "xAB" (nanabanana$)

Suppose "xA" (nana) is a prefix in the tree

Then "A" (ana) is a prefix in the tree

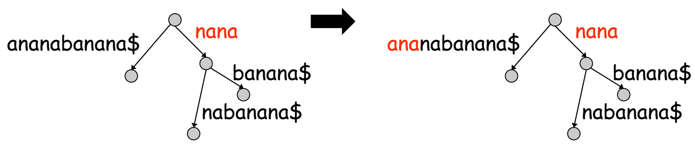
⇒ Don't need to match A again — assume it's there.

S = nananabanana$

↑ i ↑ j : end of match

S = nananabanana$

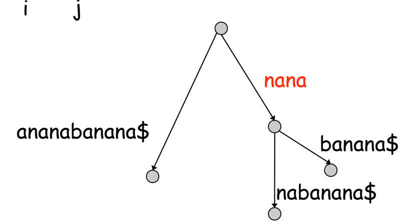
↑ i ↑ j



Suffix Tree Construction

S = nananabanana$

↑ i ↑ j

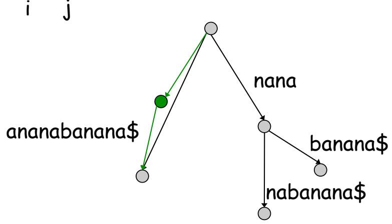


inserting "nanabanana$" matches "nana": 4 comparisons

Suffix Tree Construction

S = nananabanana$

↑ i ↑ j

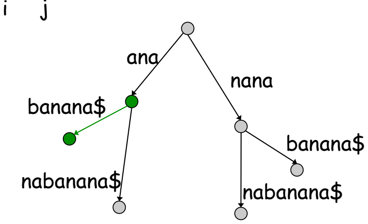


inserting "anabanana$", "ana" already matched.
just follow "a…" edge and split : 0 comparisons

Suffix Tree Construction

S = nananabanana$

↑ i ↑ j

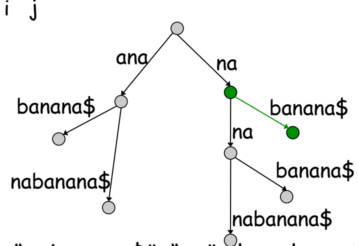


inserting "anabanana$", "ana" already matched.
just follow "a…" edge and split : 0 comparisons

Suffix Tree Construction

S = nananabanana$

i j

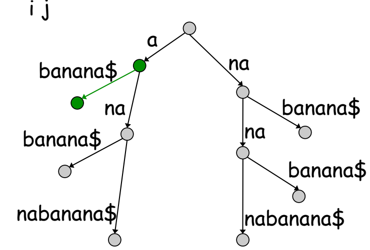


inserting “nabanana$”, “na” already matched.
just follow “n…” edge and split : 0 comparisons

Suffix Tree Construction

S = nananabanana$

i j

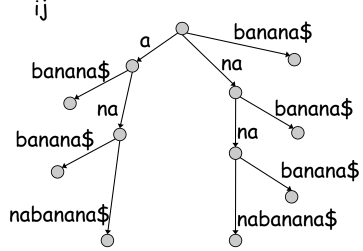


inserting “abanana$”, “a” already matched.
just follow “a…” edge and split : 0 comparisons

Suffix Tree Construction

S = nananabanana$

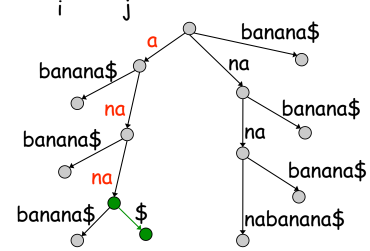
ij



Suffix Tree Construction

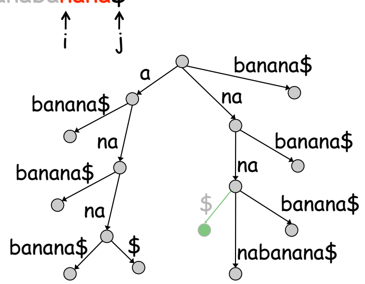
S = nananabanana$

i j



Suffix Tree Construction

S = nananabanana$

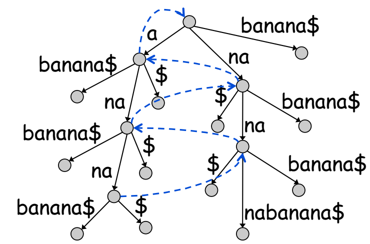


Uh oh. Yes, "nana" is already there, but it takes extra work to follow intermediate nodes

Second Improvement: Suffix Links

For every internal node "xA", keep a pointer to the node for "A"

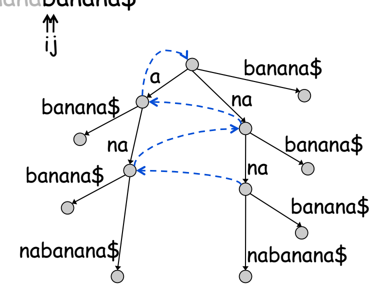
S = nananabanana$



Getting from xA to A becomes cheaper — follow suffix links!

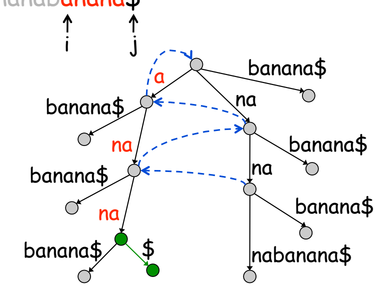
Suffix Tree Construction

S = nananabanana$



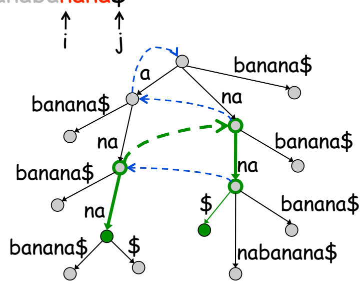
Suffix Tree Construction

S = nananabanana$



Suffix Tree Construction

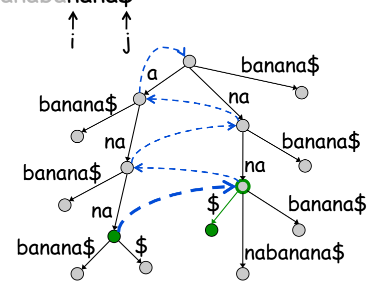
S = nananabanana$



Follow nearest (parent of new node's) suffix link to get most of the way to "nana"

Suffix Tree Construction

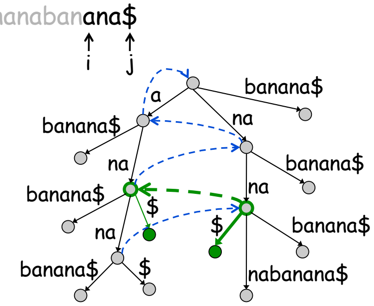
S = nananabanana$



After inserting "nana$", "nana" internal node exists.
Connect "anana" to "nana"

Suffix Tree Construction

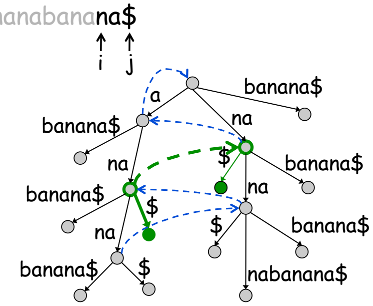
S = nananabanana$



Follow nearest suffix link to get from "nana" to "ana"

Suffix Tree Construction

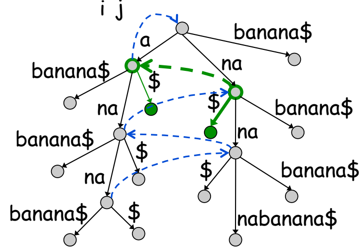
S = nananabanana$



Suffix Tree Construction

S = nananabanana$

↑↑
i j



Suffix Tree Construction

S = nananabanana$

↑ ↑
i j

Algorithm:

for i=1 to n

1. Search for S[i..j-1], incrementing j until no match, i.e., S[i..j-1] in tree but not S[i..j]
2. **if** search is in middle of an edge:

   then split edge at S[i..j-1]

   add new child of S[i..j-1] with suffix S[j..n]
3. Use parent's suffix link to find S[i+1..j-1] and split edge here if not already split
4. If split edge in (2), add suffix link from S[i..j-1] to S[i+1..j-1]

Almost Correct Analysis

Each increment of j takes O(1) time

- Just search one more character

Each iteration of i takes O(1) additional time

- Just following suffix link (or starting from root) to get from S[i..j-1] to S[i+1..j-1]

⇒ total time is O(n) because i and j are each incremented n times.

What's wrong with this argument?

Following Suffix Links

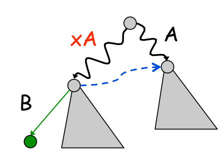
Suppose S[i..j-1] node is **not** a new node,

then suffix link from S[i..j-1] to S[i+1..j-1] already exists,

so find S[i+1..j-1] in O(1) time.

S = ZxAB

↑ ↑
i j



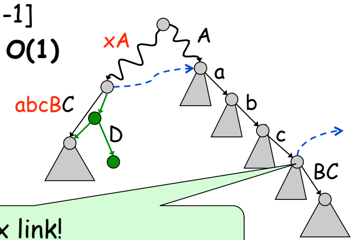
Following Suffix Links

Suppose S[i..j-1] **is** a new node (*i.e.*, resulting from an edge split)

1. Go to parent node S[i..k], for some i ≤ k < j-1
2. Follow suffix link to S[i+1..k]
3. Search down to S[i+1..j-1]

   not guaranteed to be O(1)

S = ZxAabcBD
↑ i ↑ j



This node has a suffix link!
Thus, k must increase in next iteration!

Better Analysis

S =

current suffix ↑ i ↑ k nearest suffix link ↑ j matched prefix

(k not maintained by algorithm)

j: Each increment of j takes O(1) time

k: If searching from S[i+1..k] to S[i+1..j-1] follows r edges, then k increases by at least r

i: Each iteration of i takes O(1) additional time

⇒ total time is O(n) because i, j, and k are each incremented at most n times.

Extending to multiple lists

Suppose we want to match a pattern with a dictionary of m texts with a total length of n

Concatenate all texts (interspersed with special characters) and construct a common suffix tree

(Optional) truncate tree below special characters

Time : O(n+m) = O(n)

or

Construct suffix tree on first text, then insert suffixes of second text and so on.

Time: also O(n)

Longest Common Substring

Find the longest string that is a substring of both S_1 and S_2

Construct a common suffix tree for both texts

All leaves should be labeled with S_1 and/or S_2

Any node with a descendent labeled S_1 and S_2 is a common substring

The "deepest" such node is the longest common substring

Takes linear time with a tree traversal

Common substrings of m texts

Given m texts of total length n

Given some value $1 \le k \le m$, find the longest string that is a substring of at least k texts

Construct a common suffix tree, labeling each leaf with the text it comes from

For every internal node, find the number of distinctly labeled descendents : $O(m)$ time per node.

Can report for all k with a single tree traversal

time : $O(mn)$ — not linear