15-853:Algorithms in the Real World

Linear and Integer Programming I
- Introduction
- Geometric Interpretation
- Simplex Method

Linear and Integer Programming

Linear or Integer programming

maximize $z = c^T x$ **cost** or **objective** function
subject to $Ax = b$ **equalities**
$x \geq 0$ **inequalities**
$c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

Linear programming:

$x \in \mathbb{R}^n$ (polynomial time)

Integer programming:

$x \in \mathbb{Z}^n$ (NP-complete)

Extremely general framework, especially IP

Related Optimization Problems

Unconstrained optimization

$\max\{f(x) : x \in \mathbb{R}^n\}$

Constrained optimization

$\max\{f(x) : g_i(x) \leq 0, h_j(x) = 0\}$

Quadratic programming

$\max\{1/2x^TQx + c^Tx : Ax \leq b, Ex = d\}$

Zero-One programming

$\max\{c^Tx : Ax = b, x \in \{0,1\}^n, c \in \mathbb{R}^n, b \in \mathbb{R}^m\}$

Mixed Integer Programming

$\max\{c^Tx : Ax = b, x \geq 0, x_i \in \mathbb{Z}^n, i \in I, x_r \in \mathbb{R}^n, r \in R\}$

How important is optimization?

- 50+ packages available
- 1300+ papers just on interior-point methods
- 100+ books in the library
- 10+ courses at most Universities
- 100s of companies
- All major airlines, delivery companies, trucking companies, manufacturers, … make serious use of optimization.

Linear+Integer Programming Outline

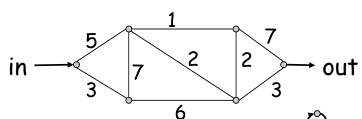
Linear Programming
- General formulation and geometric interpretation
- Simplex method
- Ellipsoid method
- Interior point methods

Integer Programming
- Various reductions of NP hard problems
- Linear programming approximations
- Branch-and-bound + cutting-plane techniques

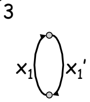
Applications of Linear Programming

1. A substep in most integer and mixed-integer linear programming (MIP) methods
2. Selecting a mix: oil mixtures, portfolio selection
3. Distribution: how much of a commodity should be distributed to different locations.
4. Allocation: how much of a resource should be allocated to different tasks
5. Network Flows

Linear Programming for Max-Flow

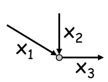


Create two variables per edge:



Create one equality per vertex:

$x_1 + x_2 + x_3' = x_1' + x_2' + x_3$



and two inequalities per edge:

$x_1 \le 3, \ x_1' \le 3$

add edge x_0 from **out** to **in**

maximize x_0

In Practice

In the "real world" most problems involve at least some integral constraints.

- Many resources are integral
- Can be used to model yes/no decisions (0-1 variables)

Therefore "1. A subset in integer or MIP programming" is the most common use in practice

Algorithms for Linear Programming

- **Simplex** (Dantzig 1947)
- **Ellipsoid** (Kachian 1979)
  first algorithm known to be **polynomial time**
- **Interior Point**
  first practical polynomial-time algorithms
  - **Projective method** (Karmakar 1984)
  - **Affine Method** (Dikin 1967)
  - **Log-Barrier Methods** (Frisch 1977, Fiacco 1968, Gill et.al. 1986)

Many of the interior point methods can be applied to nonlinear programs. Not known to be poly. time

State of the art

1 million variables

10 million nonzeros

No clear winner between Simplex and Interior Point

- Depends on the problem
- Interior point methods are subsuming more and more cases
- All major packages supply both

The truth: the sparse matrix routines, make or break both methods.

The best packages are highly sophisticated.

Comparisons, 1994

problem	Simplex (primal)	Simplex (dual)	Barrier + crossover
binpacking	29.5	62.8	560.6
distribution	18,568.0	won't run	too big
forestry	1,354.2	1,911.4	2,348.0
maintenace	57,916.3	89,890.9	3,240.8
crew	7,182.6	16,172.2	1,264.2
airfleet	71,292.5	108,015.0	37,627.3
energy	3,091.1	1,943.8	858.0
4color	45,870.2	won't run	too big

Formulations

There are many ways to formulate linear programs:

- **objective (or cost) function**
  maximize $c^T x$, or
  minimize $c^T x$, or
  find any feasible solution
- **(in)equalities**
  $Ax \le b$, or
  $Ax \ge b$, or
  $Ax = b$, or any combination
- **nonnegative variables**
  $x \ge 0$, or not

Fortunately it is pretty easy to convert among forms

Formulations

The two **most common** formulations:

Canonical form	slack variables	Standard form
maximize $c^T x$ subject to $Ax \le b$ $x \ge 0$	→	maximize $c^T x$ subject to $Ax = b$ $x \ge 0$

e.g.

$7x_1 + 5x_2 \le 7$ $x_1, x_2 \ge 0$	y_1 →	$7x_1 + 5x_2 + y_1 = 7$ $x_1, x_2, y_1 \ge 0$

More on slack variables later.

Geometric View

A **polytope** in n-dimensional space

Each inequality corresponds to a half-space.

The "feasible set" is the intersection of the half-spaces

This corresponds to a polytope

Polytopes are **convex**: if x,y is in the polytope, so is the line segment joining them.

The optimal solution is at a vertex (i.e., a corner).

Simplex moves around on the surface of the polytope

Interior-Point methods move within the polytope

Geometric View

maximize:

$z = 2x_1 + 3x_2$

subject to:

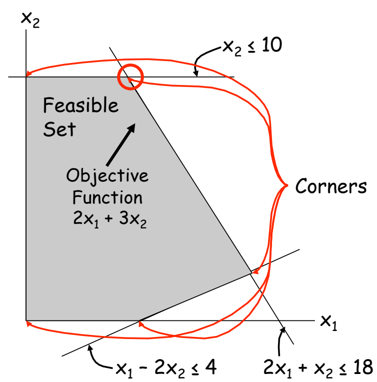
$x_1 - 2x_2 \le 4$

$2x_1 + x_2 \le 18$

$x_2 \le 10$

$x_1, x_2 \ge 0$

An intersection of 5 halfspaces



Notes about higher dimensions

For n dimensions and no degeneracy

Each corner (extreme point) consists of:

- n intersecting (n-1)-dimensional **hyperplanes**
  e.g. for n = 3, 3 intersecting 2d planes make corner
- n intersecting **edges**

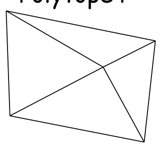
Each edge corresponds to moving off of one hyperplane (still constrained by n-1 of them)

 # Corners can be exponential in n (e.g., a hypercube)

Simplex will move from corner to corner along the edges

The Simple Essense of Simplex

Polytope P



Input: max $f(x) = cx$
s.t. x in $P = \{x : Ax \leq b, x \geq 0\}$

Consider Polytope P from canonical form as a graph $G = (V,E)$ with
V = polytope vertices,
E = polytope edges.

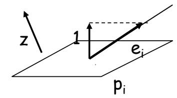
1) Find *any* vertex v of P.
2) While there exists a neighbor u of v in G with $f(u) < f(v)$, update v to u.
3) Output v.

Choice of neighbor if several u have $f(u) < f(v)$?

Optimality and Reduced Cost

The **Optimal** solution must include a corner.

The **Reduced cost** for a hyperplane at a corner is the cost of moving one unit away from the plane along its corresponding edge.

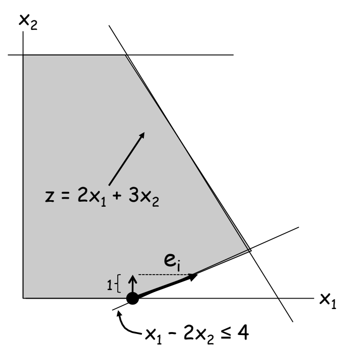


$r_i = -z \cdot e_i$

For **maximization**, if all reduced cost are non-negative, then we are at an optimal solution.

Finding the most negative reduced cost is one often used heuristic for choosing an edge to leave on

Reduced cost example



$e_i = (2, 1)$

Ex: reduced cost for leaving x_1-axis from point (4,0)

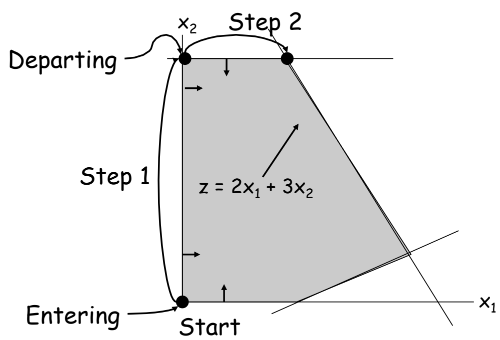
Moving 1 unit off of x_1-axis will move us (2,1) units along the edge.

The reduced cost of leaving the plane x_1 is $-(2,3) \cdot (2,1) = -7$

Simplex Algorithm

1. Find a **corner of the feasible region**
2. **Repeat**
   A. For each of the n hyperplanes intersecting at the corner, calculate its **reduced cost**
   B. If they are all non-negative, then **done**
   C. Else, pick the most negative reduced cost
   This is called the **entering** plane
   D. Move along corresponding edge (i.e. leave that hyperplane) until we reach the next corner (i.e. reach another hyperplane)
   The new plane is called the **departing** plane

Example



Simplifying

Problem:

- The $Ax \leq b$ constraints not symmetric with the $x \geq 0$ constraints.
  We would like more symmetry.

Idea:

- Leave only inequalities of the form $x \geq 0$.

Use "slack variables" to do this.

Convert into form:

maximize $c^T x$
subject to $Ax = b$
$x \geq 0$

Standard Form

maximize $c^T x$
subject to $Ax \leq b$
$x \geq 0$

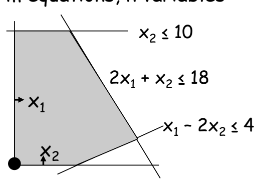
slack variables →

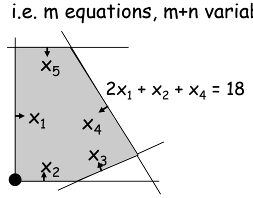
Standard Form

maximize $c^T x'$
subject to $A'x' = b$
$x' \geq 0$

$|A| = m \times n$
i.e. m equations, n variables

$|A'| = m \times (m+n)$
i.e. m equations, m+n variables





Example, again

maximize:

$z = 2x_1 + 3x_2$

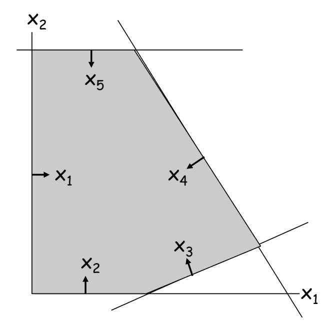
subject to:

$x_1 - 2x_2 + \underline{x_3} = 4$

$2x_1 + x_2 + \underline{x_4} = 18$

$x_2 + \underline{x_5} = 10$

$x_1, x_2, \underline{x_3}, \underline{x_4}, \underline{x_5} \geq 0$

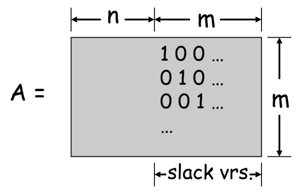


The equality constraints impose a 2d plane embedded in 5d space, looking at the plane gives the figure above

Using Matrices

If before adding the slack variables A has size m × n then after it has size m × (n + m)

m can be larger or smaller than n

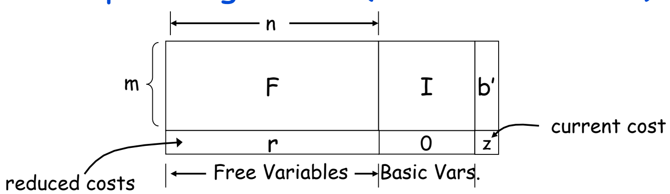


Assuming rows are independent, the solution space of Ax = b is an n-dimensional subspace.

Simplex Algorithm, again

1. Find a **corner of the feasible region**
2. **Repeat**
   A. For each of the n hyperplanes intersecting at the corner, calculate its **reduced cost**
   B. If they are all non-negative, then **done**
   C. Else, pick the most negative reduced cost
   This is called the **entering** plane
   D. Move along corresponding line (i.e. leave that hyperplane) until we reach the next corner (i.e. reach another hyperplane)
   The new plane is called the **departing** plane

Simplex Algorithm (Tableau Method)



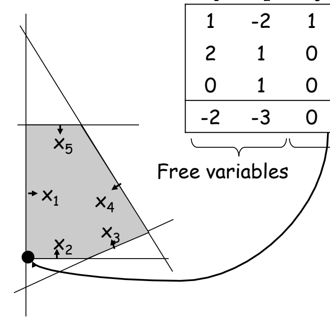
This form is called a **Basic Solution**
- the n "**free**" variables are set to 0
- the m "**basic**" variables are set to b'

A valid solution to Ax = b if reached using Gaussian Elimination
Represents n intersecting hyperplanes
If feasible (i.e. b' ≥ 0), then the solution is called a **Basic Feasible Solution** and is a corner of the feasible set

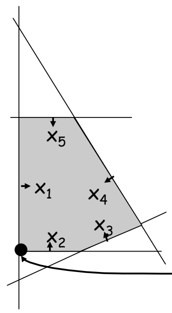
Corner

x_1	x_2	x_3	x_4	x_5		
1	-2	1	0	0	4	$x_1 - 2x_2 + x_3 = 4$
2	1	0	1	0	18	$2x_1 + x_2 + x_4 = 18$
0	1	0	0	1	10	$x_2 + x_5 = 10$
-2	-3	0	0	0	0	$z - 2x_1 - 3x_2 = 0$

Free variables (x1, x2) — Basic Vars. (x3, x4, x5)



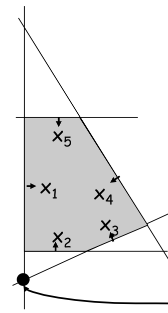
Corner



free variables / basic variables

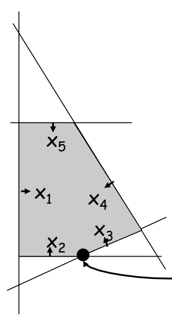
1	-2	1	0	0	4
2	1	0	1	0	18
0	1	0	0	1	10
-2	-3	0	0	0	0
x_1	x_2	x_3	x_4	x_5	

Corner



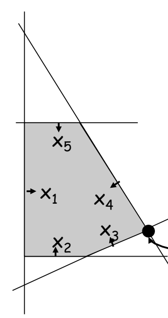
-.5	-.5	1	0	0	-2
2.5	.5	0	1	0	20
.5	.5	0	0	1	12
-3.5	-1.5	0	0	0	-6
x_1	x_3	x_2	x_4	x_5	

Corner



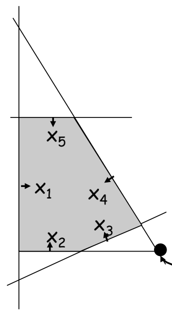
1	-2	1	0	0	4
-2	5	0	1	0	10
0	1	0	0	1	10
2	-7	0	0	0	8
x_3	x_2	x_1	x_4	x_5	

Corner



.2	.4	1	0	0	8
-.4	.2	0	1	0	2
.4	-.2	0	0	1	8
-.8	1.4	0	0	0	22
x_3	x_4	x_1	x_2	x_5	

Corner



-.5	-2.5	1	0	0	-5
.5	.5	0	1	0	9
0	1	0	0	1	10
1	-2	0	0	0	18
x_4	x_2	x_3	x_1	x_5	

Note that in general there are n+m choose m corners

Simplex Method Again

Once you have found a basic feasible solution (a corner), we can move from corner to corner by swapping columns and eliminating.

ALGORITHM

1. Find a **basic feasible solution**
2. **Repeat**
   A. If r (reduced cost) ≥ 0 , DONE
   B. Else, pick column with most negative r
   C. Pick row with least positive b'/(selected column)
   D. Swap columns
   E. Use Gaussian elimination to restore form

Tableau Method

A. If r are all non-negative then **done**

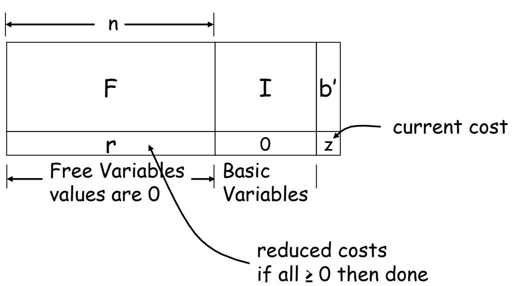


Tableau Method

B. Else, pick the most negative reduced cost
This is called the **entering** plane

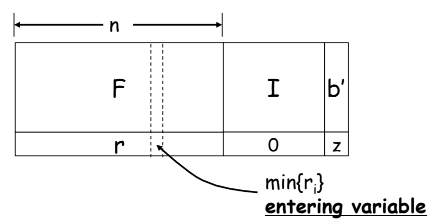


Tableau Method

C. Move along corresponding line (i.e. leave that hyperplane) until we reach the next corner (i.e. reach another hyperplane)
The new plane is called the **departing** plane

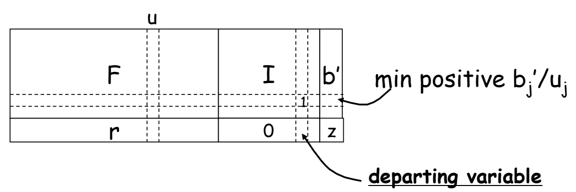
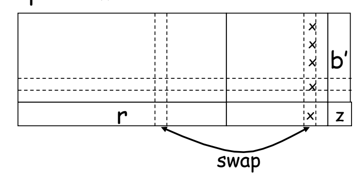


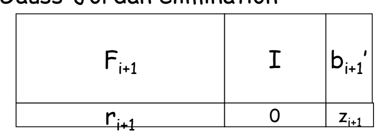
Tableau Method

D. Swap columns



No longer in proper form

E. Gauss-Jordan elimination



Back to proper form

Example

x_1	x_2	x_3	x_4	x_5	
1	-2	1	0	0	4
2	1	0	1	0	18
0	1	0	0	1	10
-2	-3	0	0	0	0

$x_1 - 2x_2 + x_3 = 4$

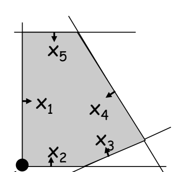
$2x_1 + x_2 + x_4 = 18$

$x_2 + x_5 = 10$

$z - 2x_1 - 3x_2 = 0$

↓ Find corner

1	-2	1	0	0	4
2	1	0	1	0	18
0	1	0	0	1	10
-2	-3	0	0	0	0
x_1	x_2	x_3	x_4	x_5	



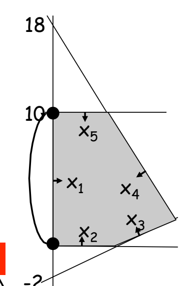
$x_1 = x_2 = 0$ (start)

Example

1	-2	1	0	0	4
2	1	0	1	0	18
0	1	0	0	1	10
-2	-3	0	0	0	0
x_1	x_2	x_3	x_4	x_5	

1	-2	1	0	0	4	-2
2	1	0	1	0	18	18
0	1	0	0	1	10	10
-2	-3	0	0	0	0	
x_1	x_2	x_3	x_4	x_5		b_j/v_j

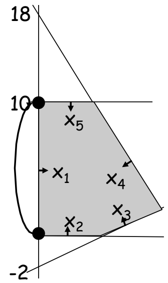
min positive



Example

swap

1	0	1	0	-2	4
2	0	0	1	1	18
0	1	0	0	1	10
-2	0	0	0	-3	0
x_1	x_5	x_3	x_4	x_2	

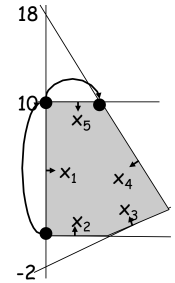


Gauss-Jordan Elimination

1	2	1	0	0	24
2	-1	0	1	0	8
0	1	0	0	1	10
-2	3	0	0	0	30
x_1	x_5	x_3	x_4	x_2	

Example

1	2	1	0	0	24
2	-1	0	1	0	8
0	1	0	0	1	10
-2	3	0	0	0	30
x_1	x_5	x_3	x_4	x_2	

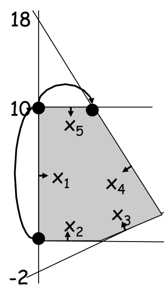


1	2	1	0	0	24	24
2	-1	0	1	0	8	4
0	1	0	0	1	10	∞
-2	-3	0	0	0	30	
x_1	x_5	x_3	x_4	x_2		

Example

swap

0	2	1	1	0	24
1	-1	0	2	0	8
0	1	0	0	1	10
0	3	0	-2	0	30
x_4	x_5	x_3	x_1	x_2	



Gauss-Jordan Elimination

-.5	2.5	1	0	0	20
.5	-.5	0	1	0	4
0	1	0	0	1	10
1	2	0	0	0	38
x_4	x_5	x_3	x_1	x_2	

Simplex Concluding remarks

For dense matrices, takes O(n(n+m)) time per iteration

Can take an **exponential** number of iterations.

In practice, sparse methods are used for the iterations.

Duality

Primal (P):

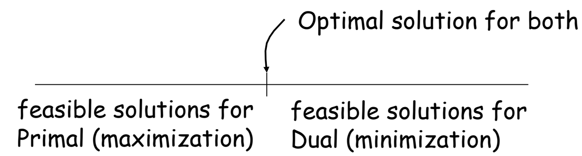
maximize $z = c^T x$
subject to $Ax \le b$
$x \ge 0$ (n equations, m variables)

Dual (D):

minimize $z = y^T b$
subject to $A^t y \ge c$
$y \ge 0$ (m equations, n variables)

Duality Theorem: if x is feasible for **P** and y is feasible for **D**, then $cx \le yb$ and at optimality $cx = yb$.

Duality (cont.)



Quite similar to duality of Maximum Flow and Minimum Cut.

Useful in many situations.

Duality Example

Primal:

maximize:

$z = 2x_1 + 3x_2$

subject to:

$x_1 - 2x_2 \le 4$
$2x_1 + x_2 \le 18$
$x_2 \le 10$
$x_1, x_2 \ge 0$

Dual:

minimize:

$z = 4y_1 + 18y_2 + 10y_3$

subject to:

$y_1 + 2y_2 \ge 2$
$-2y_1 + y_2 + y_3 \ge 3$
$y_1, y_2, y_3 \ge 0$

Solution to both is 38 (x_1=4, x_2=10), (y_1=0, y_2=1, y_3=2).