15-853:Algorithms in the Real World

String Searching I
- Tries, Patricia trees
- Suffix trees

Exact string searching

Given a text T of length n and pattern P of length m

"Quickly" find an occurrence (or all occurrences) of P in T

A Naïve solution:

Compare P with T[i…i+m] for all i --- O(nm) time

How about O(n+m) time? (Knuth Morris Pratt)

How about O(n) preprocessing time and O(m) search time?

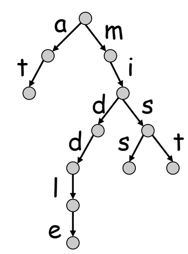
Notation:

Capital letters for strings : A, B, S

Lower case letters for characters : a, b, c, x, y, …

TRIEs

Dictionary = {at, middle, miss, mist}

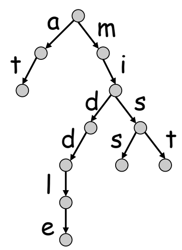


TRIEs (searching)

Consider an alphabet ∑, with |∑| = k

Assume a total of n nodes in the trie.

Consider searching a string of length m to see if it is a **prefix** of an element in the dictionary.



Search("mid",T)

TRIEs (searching)

Consider an alphabet ∑, with |∑| = k

Total of n nodes in trie.

Consider searching a string of length m to see if it is a **prefix** of an element in the dictionary.

Implementation choices:

- Array per node: O(nk) space, O(m) time search
- Tree per node: O(n) space, O(m log k) time search
- Hash children: O(n) space, O(m) time
  can hash node pointer and child character

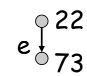


Table.Lookup((22,e)) = 73

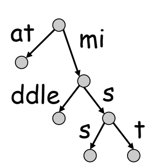
PATRICIA Trees

PATRICIA: *Practical Algorithm to Retrieve Information Coded in Alphanumeric* (1968)

Also called radix trees or compressed TRIEs

All nodes with single child are collapsed.

Dictionary = {at, middle, miss, mist}



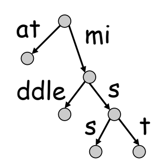
Takes less space in practice

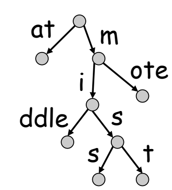
Insertion

Inserting string S into a PATRICIA tree

- Find longest common prefix
- Split edge if needed
- Add suffix

Insert("mote",T)





Takes O(|S|) time

Using Suffixes

If we want to search for any substring within a string we can store all suffixes of the string in a TRIE or PATRICIA tree.

S = mississippi

Dictionary =

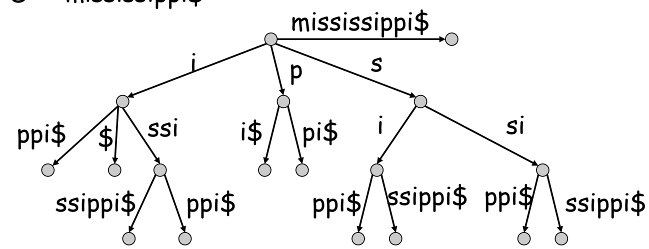
{mississippi, ississippi, ssissippi, sissippi, issippi, ssippi, sippi, ippi, ppi, pi, i}

Typically use special character ($) at the end of a string to make sure every entry has its own leaf

Suffix Trees

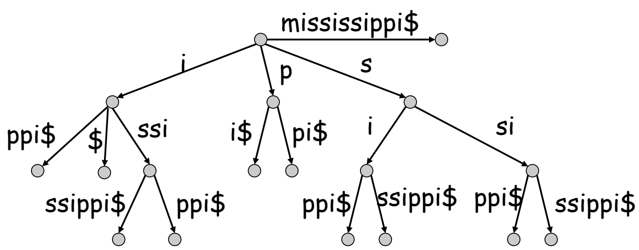
Patricia tree on all suffixes of a string.

S = "mississippi$"



Suffix Tree Space

How do we store a suffix tree in O(n) space?



Suffix Tree Construction

Simple algorithm:

T = empty
for i = 1 to n
  insert(S[i:n],T)

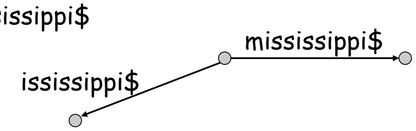
Takes $O(n^2)$ time.

Suffix Tree Construction

mississippi$

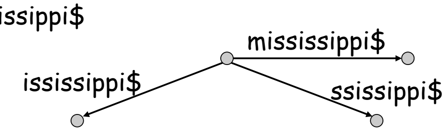
Suffix Tree Construction

ississippi$



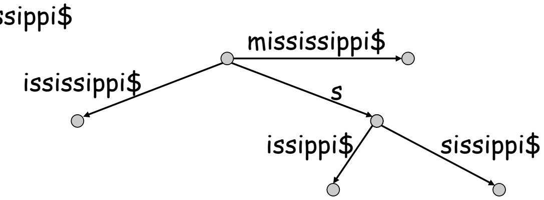
Suffix Tree Construction

ssissippi$



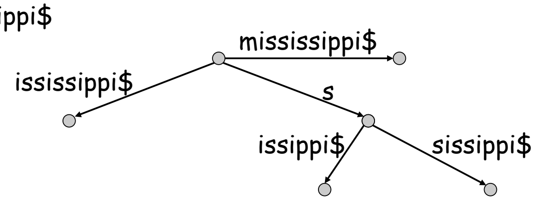
Suffix Tree Construction

sissippi$



Suffix Tree Construction

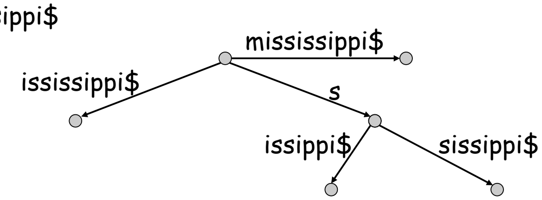
issippi$



When we look up "issi" can we make looking up "ssi" for the next step cheaper?

Suffix Tree Construction

issippi$



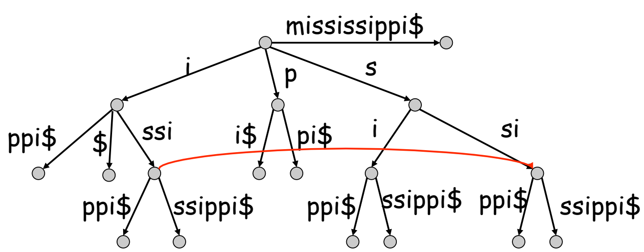
i s s i p p i $

When we previously "looked up" "issi" didn't we then also look up "ssi", "si", "s" on later steps

Suffix Links

For every internal node for a string "aS", keep a pointer to the node for "S"

Why must it exist?



Suffix Links

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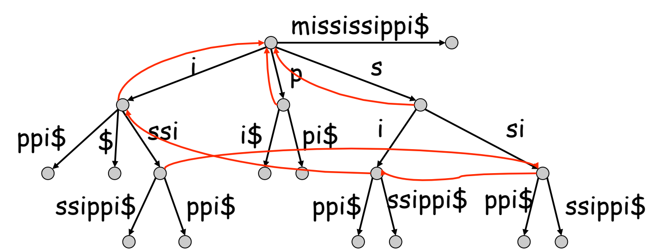
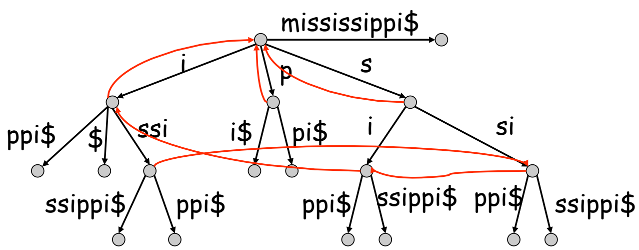
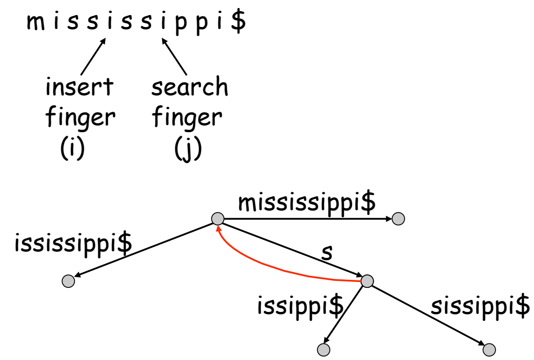


Suffix Links

Now if I have found "issi" finding "ssi" is easy, and then finding "si".



Suffix Tree Construction



Suffix Tree Construction

m i s s i s s i p p i $

↑ i ↑ j

Algorithm:

Repeat from i = 1 until i == n

1. Search from S[i:j-1] incrementing j until no match.
   i.e. found S[i:j-1] in tree but not S[i:j]
2. If search is in the middle of an edge:
   Then split edge at S[i:j-1] and add suffix S[j:n]
   Else add new child to S[i:j-1] with suffix S[j:n]
3. Use parent's suffix link to find S[i+1:j-1] and split edge here if not already split.
4. If split edge in 2, add suffix link from S[i:j-1] to S[i+1:j-1]
5. i = i + 1

Almost Correct Analysis

Each increment of j takes O(1) time
- Just search one more character

Each increment of i takes O(1) time
- Just follow suffix link

Total time is O(n) since i and j are each incremented O(n) times.

What is wrong?

Suffix Tree Construction

m i s s i s s i p p i $
↑ ↑
i j

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5. i = i + 1

Following Suffix Links

1. Go to parent of edge that is being split
   - S[i:k] for some i ≤ k < j
2. Follow link to S[i+1:k]
3. Search down for S[i+1:j-1]
   - This step might not be O(1) time
4. Now k = j (charge searching to incrementing k)

Following Suffix Links

1. Go to parent of edge that is being split
   - S[i:k] for some i ≤ k ≤ j
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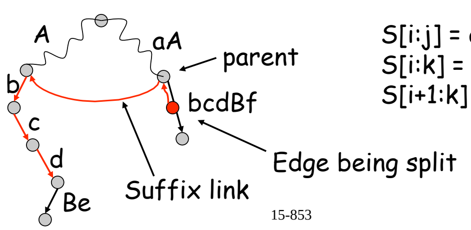
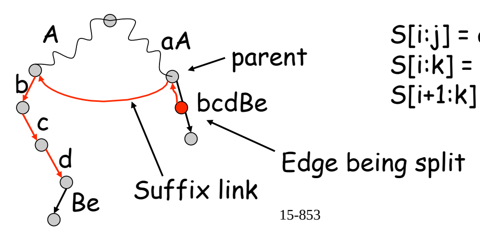


S[i:j] = aAbcdBe
S[i:k] = aA
S[i+1:k] = A

Following Suffix Links

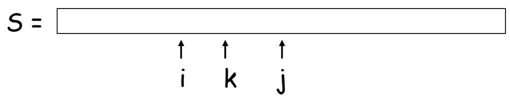
Note that searching edge Be to find B takes constant time even if B is long.

Why?



S[i:j] = aAbcdBe
S[i:k] = aA
S[i+1:k] = A

The “Three Finger” Analysis

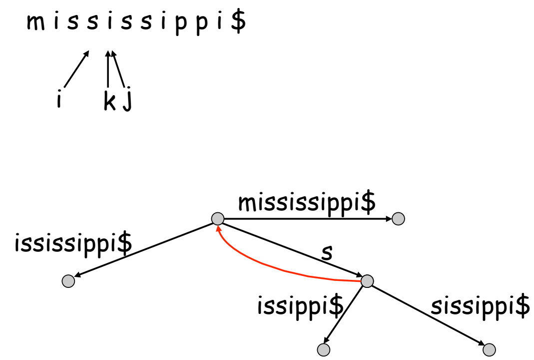


Note: there is no counter for k, it is the location of the next node up (inclusive) of S[i:j-1] in the search

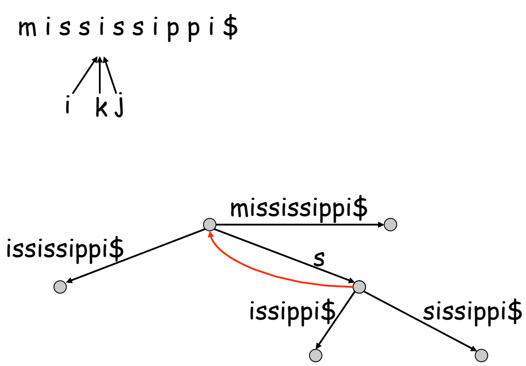
Each increment of j takes O(1) time
Following suffix link to increment i takes O(1) time
Each “increment” of k to find S[i+1:j-1] takes O(1) time

TOTAL TIME = O(n)

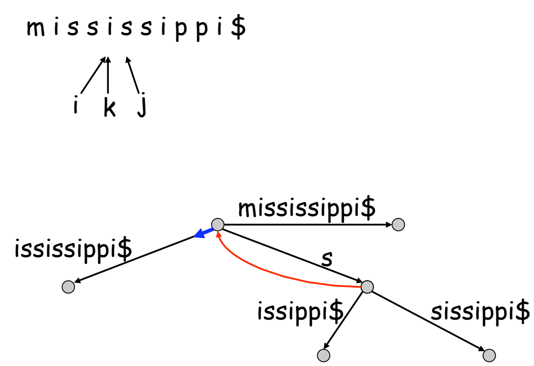
Suffix Tree Construction



Suffix Tree Construction

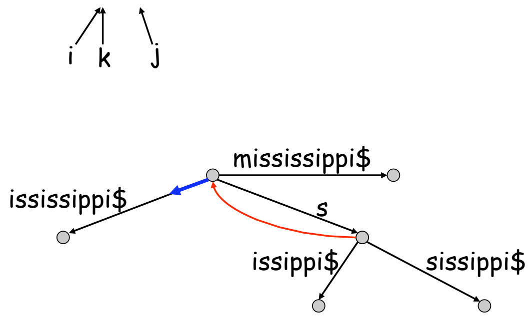


Suffix Tree Construction



Suffix Tree Construction

m i s s i s s i p p i $



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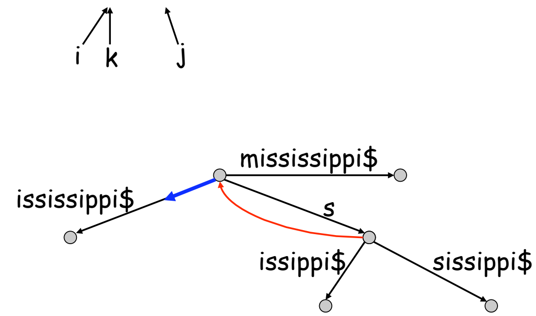
Suffix Tree Construction

m i s s i s s i p p i $



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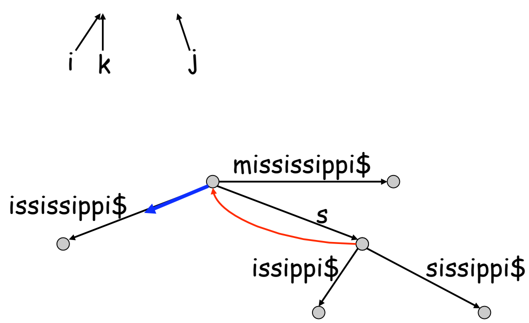
Suffix Tree Construction

m i s s i s s i p p i $



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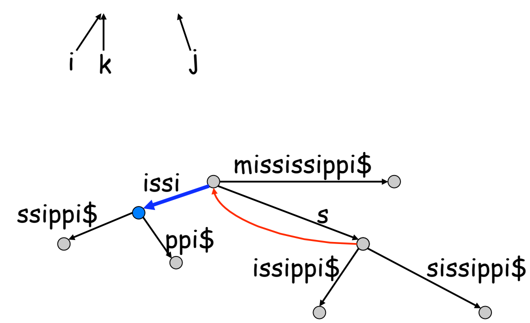
Suffix Tree Construction

m i s s i s s i p p i $



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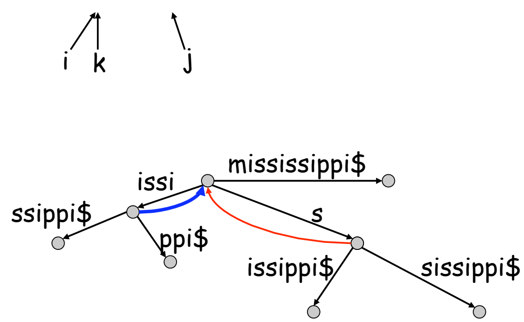
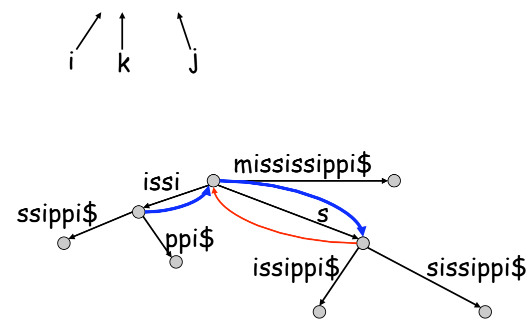
Suffix Tree Construction

m i s s i s s i p p i $



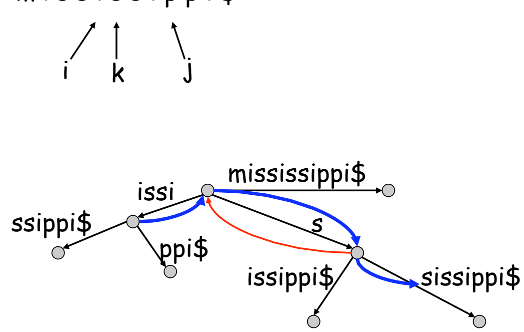
Suffix Tree Construction

m i s s i s s i p p i $



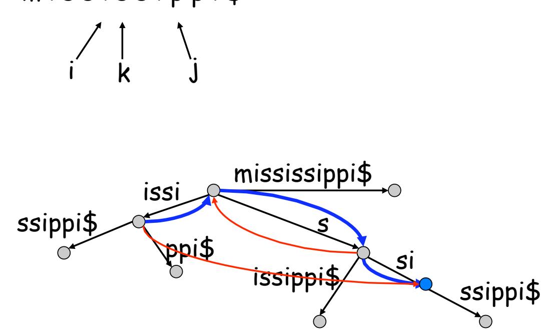
Suffix Tree Construction

m i s s i s s i p p i $

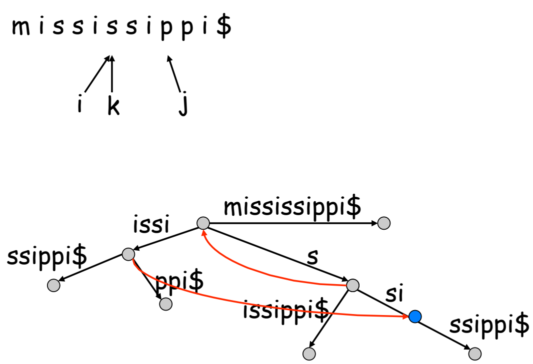


Suffix Tree Construction

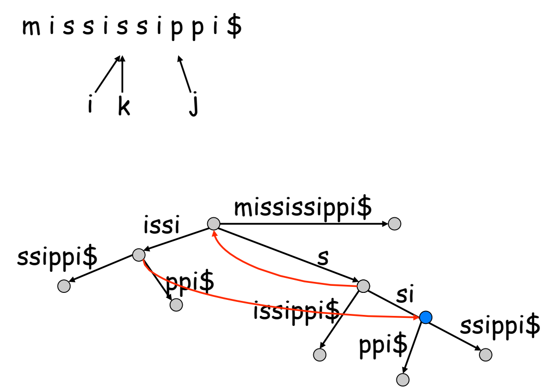
m i s s i s s i p p i $



Suffix Tree Construction



Suffix Tree Construction



Summary

Really the only change over the naïve $O(n^2)$ algorithm is the use of suffix links to speed up search when inserting each suffix.

i.e. the key is linking S[i:j] to S[i+1:j] and just doing this for internal nodes in the tree is sufficient.

Suffix trees have many applications beyond string searching.

Extending to multiple lists

Suppose we want to match a pattern with a dictionary of k strings with a total length m.

Concatenate all the strings (interspersed with special characters) and construct a common suffix tree

Time taken = $O(m + k)$

Unnecessarily complicated tree; needs special characters

Multiple lists – Better algorithm

First construct a suffix tree on the first string, then insert suffixes of the second string and so on

Each leaf node should store values corresponding to each string

O(m) as before

Longest Common Substring

Find the longest string that is a substring of both S_1 and S_2

Construct a common suffix tree for both

Any node that has descendants labeled with S_1 and S_2 indicates a common substring

The "deepest" such node is the required substring

Can be found in linear time by a tree traversal.

Common substrings of M strings

Given M strings of total length n, find for every k, the length l_k of the longest string that is a substring of at least k of the strings

Construct a common suffix tree labeling each leaf with the string it came from

For every internal node, find the number of distinctly labeled decendants

Report l_k by a single tree traversal

O(Mn) time – not linear!

Lempel-Ziv compression

Recall that at each stage, we output a pair (p_i, l_i) where $S[p_i .. p_i+l_i] = S[i .. i+l_i]$

Find all pairs (p_i,l_i) in linear time

Construct a suffix tree for S

Label each internal node with the minimum of labels of all leaves below it – this is the first place in S where it occurs. Call this label c_v.

For every i, search for the string S[i .. m] stopping just before c_v, i. This gives us l_i and p_i.