15-853:Algorithms in the Real World

Error Correcting Codes III (expander based codes)

- Expander graphs
- Low density parity check (LDPC) codes
- Tornado codes

Thanks to Shuchi Chawla for many of the slides

Why Expander Based Codes?

Linear codes like RS & random linear codes

The other two give nearly optimal rates
But they are slow :

Code	Encoding	Decoding*
Random Linear	$O(n^2)$	$O(n^3)$
RS	$O(n \log n)$	$O(n^2)$
LDPC	$O(n^2)$ or better	$O(n)$
Tornado	$O(n \log 1/\varepsilon)$	$O(n \log 1/\varepsilon)$

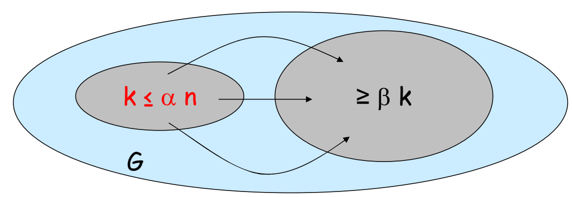
Assuming an $(n, (1-p)n, (1-\varepsilon)pn+1)_2$ tornado code
*does not necessarily fix (d-1)/2 errors

Error Correcting Codes Outline

Introduction
Linear codes
Read Solomon Codes
Expander Based Codes

- Expander Graphs
- Low Density Parity Check (LDPC) codes
- Tornado Codes

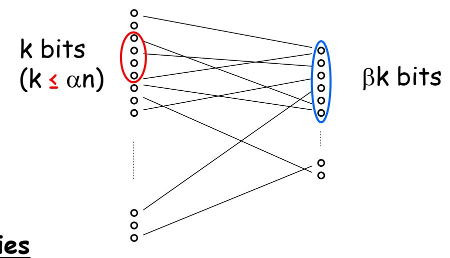
Expander Graphs (non-bipartite)



Properties

- **Expansion:** every small subset ($k \le \alpha n$) has many ($\ge \beta k$) neighbors
- **Low degree** – not technically part of the definition, but typically assumed

Expander Graphs (bipartite)



Properties

- **Expansion:** every small subset ($k \le \alpha n$) on left has many ($\ge \beta k$) neighbors on right
- **Low degree** – not technically part of the definition, but typically assumed

Expander Graphs

Useful properties:

- Every set of vertices has many neighbors
- Every balanced cut has many edges crossing it
- A random walk will quickly converge to the stationary distribution (rapid mixing)
- The graph has "high dimension"
- Expansion is related to the eigenvalues of the adjacency matrix

Expander Graphs: Applications

Pseudo-randomness: implement randomized algorithms with few random bits

Cryptography: strong one-way functions from weak ones.

Hashing: efficient n-wise independent hash functions

Random walks: quickly spreading probability as you walk through a graph

Error Correcting Codes: several constructions

Communication networks: fault tolerance, gossip-based protocols, peer-to-peer networks

d-regular graphs

An undirected graph is **d-regular** if every vertex has d neighbors.

A bipartite graph is **d-regular** if every vertex on the left has d neighbors on the right.

The constructions we will be looking at are all d-regular.

Expander Graphs: Eigenvalues

Consider the normalized adjacency matrix A_{ij} for an undirected graph G (all rows sum to 1)

The (x_i, λ_i) satisfying

$A x_i = \lambda_i x_i$

are the **eigenvectors** (x_i) and **eigenvalues** (λ_i) of A.

Consider the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \ldots$

For a d-regular graph, $\lambda_0 = 1$. Why?

The separation of the eigenvalues tell you a lot about the graph (we will revisit this several times).

If λ_1 is much smaller than λ_0 then the graph is an expander.

Expansion $\beta \geq (1/\lambda_1)^2$

Expander Graphs: Constructions

Important parameters: size (n), degree (d), expansion (β)

Randomized constructions

- A random d-regular graph is an expander with a high probability
- Construct by choosing d random perfect matchings
- Time consuming and cannot be stored compactly

Explicit constructions

- Cayley graphs, Ramanujan graphs etc
- Typical technique – start with a small expander, apply operations to increase its size

Expander Graphs: Constructions

Start with a small expander, and apply operations to make it bigger while preserving expansion

Squaring

- G^2 contains edge (u,w) if G contains edges (u,v) and (v,w) for some node v
- $A' = A^2 - 1/d\ I$
- $\lambda' = \lambda^2 - 1/d$
- $d' <= d^2 - d$

Size	≡
Degree	↑
Expansion	↑

Expander Graphs: Constructions

Start with a small expander, and apply operations to make it bigger while preserving expansion

Tensor Product (Kronecker product)

- G = AxB   nodes are (a,b) $\forall\ a \in A$ and $b \in B$
- edge between (a,b) and (a',b') if A contains (a,a') and B contains (b,b')
- $n' = n_1 n_2$
- $\lambda' = \max(\lambda_1, \lambda_2)$
- $d' = d_1 d_2$

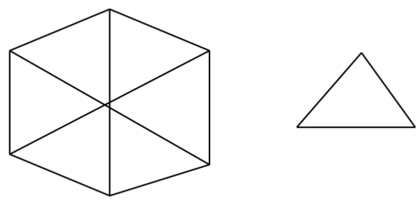
Size	↑
Degree	↑
Expansion	↓

Expander Graphs: Constructions

Start with a small expander, and apply operations to make it bigger while preserving expansion

Zig-Zag product

- "Multiply" a big graph with a small graph



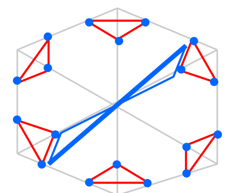
$n_2 = d_1$
$d_2 = \sqrt{d_1}$

Expander Graphs: Constructions

Start with a small expander, and apply operations to make it bigger while preserving expansion

Zig-Zag product

- "Multiply" a big graph with a small graph



Size ↑
Degree ↓
Expansion ↓ (slightly)

Combination: square and zig-zag

For a graph with size n, degree d, and eigenvalue λ, define $G = (n, d, \lambda)$. We would like to increase n while holding d and λ the same.

Squaring and zig-zag have the following effects:

$(n, d, \lambda)^2 = (n, d^2, \lambda^2)$ ≡ ↑↑

(n_1, d_1, λ_1) zz $(d_1, d_2, \lambda_2) = (n_1 d_1, d_2^2, \lambda_1 + \lambda_2 + \lambda_2^2)$ ↑↓↓

Now given a graph $H = (d^4, d, 1/5)$ and $G_1 = (d^4, d^2, 2/5)$

- $G_i = G_{i-1}^2$ zz H (square, zig-zag)

Giving: $G_i = (n_i, d^2, 2/5)$ where $n_i = d^{4i}$ (as desired)

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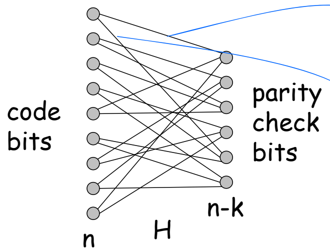
Low Density Parity Check (LDPC) Codes



n

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad n-k$$

Each row is a vertex on the right and each column is a vertex on the left.

A codeword on the left is valid if each right "parity check" vertex has parity 0.

The graph has O(n) edges (**low density**)

Applications in the "real world"

10Gbase-T (IEEE 802.3an, 2006)
- Standard for 10 Gbits/sec over copper wire

WiMax (IEEE 802.16e, 2006)
- Standard for medium-distance wireless. Approx 10Mbits/sec over 10 Kilometers.

NASA
- Proposed for all their space data systems

History

Invented by Gallager in 1963 (his PhD thesis)

Generalized by Tanner in 1981 (instead of using parity and binary codes, use other codes for "check" nodes).

Mostly forgotten by community at large until the mid 90s when revisited by Spielman, MacKay and others.

Distance of LDPC codes

Consider a d-regular LPDC with (α,3d/4) expansion.

Theorem: Distance of code is greater than αn.

Proof. (by contradiction)

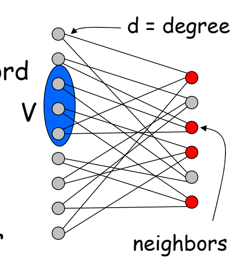
Assume a codeword with weight v ≤ αn.

Let V be the set of 1 bits in the codeword

It has >3/4dv neighbors on the right

Average # of 1s per such neighbor is < 4/3.

To make average work, at least one has only 1 bit…which would cause an error since parity has to be at least 2.



Correcting Errors in LDPC codes

We say a vertex is *unsatisfied* if parity $\neq 0$

Algorithm:

While there are unsatisfied check bits

1. Find a bit on the left for which more than d/2 neighbors are unsatisfied
2. Flip that bit

Converges since every step reduces unsatisfied nodes by at least 1.

Runs in linear time.

Why must there be a node with more than d/2 unsatisfied neighbors?

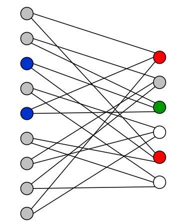
Coverges to closest codeword

Theorem: If # of error bits is less than αn/4 with 3d/4 expansion then the simple decoding algorithm will coverge to the closest codeword.

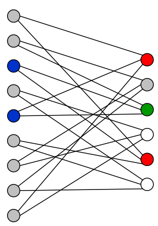
Proof: let:

- u_i = # of unsatisfied check bits on step i
- r_i = # corrupt code bits on step i
- s_i = # satisfied check bits with corrupt neighbors on step i

We know that u_i decrements on each step, but what about r_i?



Proof continued:



- u_i = unsatisfied
- r_i = corrupt
- s_i = satisfied with corrupt neighbors

$u_i + s_i \geq \frac{3}{4} d r_i$ (by expansion)

$2 s_i + u_i \leq d r_i$ (by counting edges)

$\frac{1}{2} d r_i \leq u_i$ (by substitution)

$u_i < u_0$ (steps decrease u) $u_0 \leq d r_0$ (by counting edges)

Therefore: $r_i < 2 r_0$ i.e. number of corrupt bits cannot more than double

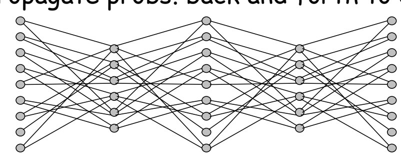
If we start with at most αn/4 corrupt bits we will never get αn/2 corrupt bits but the distance is αn

More on decoding LDPC

Simple algorithm is only guaranteed to fix half as many errors as could be fixed but in practice can do better.

Fixing (d-1)/2 errors is NP hard

Soft "decoding" as originally specified by Gallager is based on belief propagation---determine probability of each code bit being 1 and 0 and propagate probs. back and forth to check bits.



Encoding LDPC

Encoding can be done by generating G from H and using matrix multiply.

What is the problem with this?

Various more efficient methods have been studied

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The loss model

Random Erasure Model:

- Each bit is lost independently with some probability μ
- We know the positions of the lost bits

For a **rate** of (1-p) can correct $(1-\varepsilon)$p fraction of the errors.

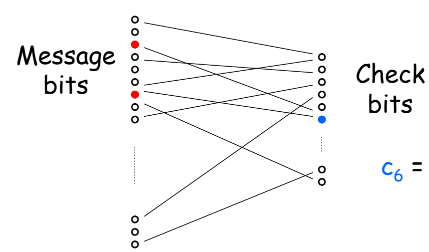
Seems to imply a

$(n, (1-p)n, (1-\varepsilon)pn+1)_2$

code, but not quite because of random errors assumption.

We will assume p = .5.

Error Correction can be done with some more effort



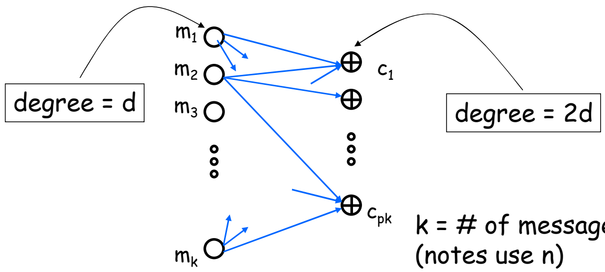
$c_6 = m_3 \oplus m_7$

Similar to LDPC codes but check bits are not required to equal zero (i.e the graph does not represent H).

Tornado codes

Will use d-regular bipartite graphs with n nodes on the left and pn on the right (notes assume p = .5)

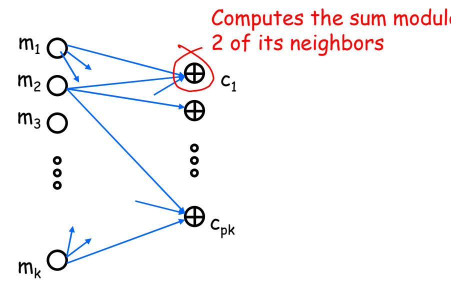
Will need β > d/2 expansion.



k = # of message bits
(notes use n)

Tornado codes: Encoding

Why is it linear time?

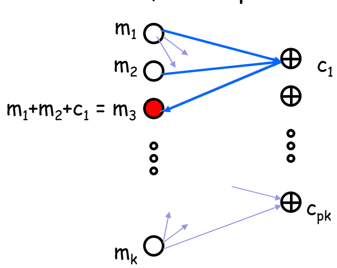


Tornado codes: Decoding

Assume that all the check bits are intact

Find a check bit such that only one of its neighbors is erased (an *unshared neighbor*)

Fix the erased code, and repeat.



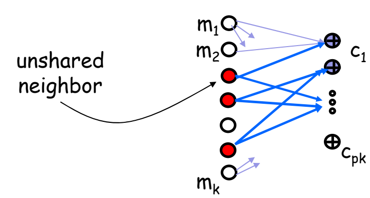
Tornado codes: Decoding

Need to ensure that we can always find such a check bit

"Unshared neighbors" property

Consider the set of corrupted message bit and their neighbors. Suppose this set is small.

=> at least one message bit has an unshared neighbor.



Tornado codes: Decoding

Can we always find unshared neighbors?

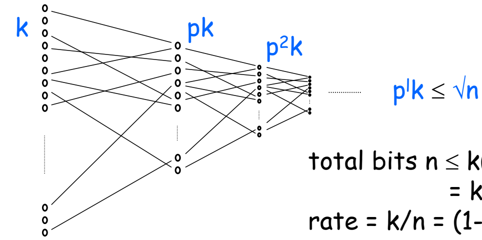
Expander graphs give us this property if $\beta > d/2$ (see notes)

Also, [Luby et al] show that if we construct the graph from a specific kind of degree sequence, then we can always find unshared neighbors.

What if check bits are lost?

Cascading

- Use another bipartite graph to construct another level of check bits for the check bits
- Final level is encoded using RS or some other code



total bits $n \le k(1 + p + p^2 + \ldots)$
$= k/(1-p)$

rate = $k/n = (1-p)$

Cascading

Encoding time

- for the first k stages : $|E| = d \times |V| = O(k)$
- for the last stage: $\sqrt{k} \times \sqrt{k} = O(k)$

Decoding time

- start from the last stage and move left
- again proportional to $|E|$
- also proportional to d, which must be at least $1/\varepsilon$ to make the decoding work

Can fix $kp(1-\varepsilon)$ random erasures

Some extra slides

Expander Graphs: Properties

Prob. Dist. – π ; Uniform dist. – u

Small $|\pi$-u$|$ indicates a large amount of "randomness"

Show that $|A\pi\text{-u}| \cdot \lambda_2|\pi\text{-u}|$

Therefore small λ_2 => fast convergence to uniform

Expansion β ¼ $(1/\lambda_2)^2$

Expander Graphs: Properties

To show that $|A\pi\text{-u}| \cdot \lambda_2|\pi\text{-u}|$

Let $\pi = u + \pi'$

u is the principle eigenvector $Au = u$

π' is perpendicular to u $A\pi' \cdot \lambda_2\pi'$

So, $A\pi \cdot u + \lambda_2\pi'$

Thus, $|A\pi - u| \cdot \lambda_2|\pi'|$