15-853:Algorithms in the Real World

Satisfiability Solvers (Lectures 1 & 2)

Satisfiability (SAT)

The “original” NP-Complete Problem.

- Input:

  Variables V = {$x_1, x_2, \ldots, x_n$}, Boolean Formula Φ (typically in conjunctive normal form (CNF)).

  e.g., Φ = (x1 v x2 v ¬x3) & (¬x1 v ¬x2 v x3) & …
- Output:

  Either a satisfying assignment *f*:V → {True, False}
  that makes Φ evaluate to True, OR
  “Unsatisfiable” if no such assignment exists.

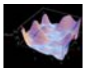
Extensions/Related Problems

- Satisfiability Modulo Theories

  Input: a formula Φ in quantifier-free first-order logic.
  Output: is Φ satisfiable?
- Theorem Provers
- Pseudo-boolean optimization
- Planners (Quantified SAT Solvers)

Applications

- Verification:
  - Hardware: Electronic design automation market is about $6 Billion
  - Protocols: e.g., use temporal logic to reason about concurrency
  - Software
- Optimization
  - Competitor to Integer Programming solutions in some domains
- Math: Prove conjectures in finite algebra



An Aside: Example Proof by Machine

- Thm: Robbins Algebra = Boolean Algebra
- Robbins Algebra: values {0,1} and 3 axioms:

  x v (y v z) = (x v y) v z
  x v y = y v z
  ¬ (¬(x v y) v ¬(x v ¬y)) = x
- Conjectured in 1933
- Proved in 1996 by prover EQP running for 8 days (RS/6000 with ~30 MB RAM)
- Limited success since 1990s.

Annual Competitions

- **SAT Competition**
- **CADE ATP System Competition**
- **ASP Solver Competition**
- **SMT-COMP**
- **Constraint Satisfaction Solver Competition**
- **Competitition/Exhibition of Termination Tools**
- **TANCS**
- **QBF Solvers Evaluation**
- Open Source Solvers: SATLIB, SATLive

Algorithms for SAT

- Complete (satisfying assignment or *UNSAT*)
  - Davis-Putnam-Logemann-Loveland algorithm (DPLL)
- Incomplete (satisfying assignment or *FAILURE*)
  - GSAT
  - WalkSAT

Prerequisite: Proof Systems

- **What constitutes a proof of unsatisfiability?**
- For a language L in {0,1}*, a **proof system** for membership in L is a poly-time computable function P such that
  - For all x in L, there is a witness y with P(x,y) = 1
  - For all x not in L, for all y, P(x,y) = 0
- Complexity: worst case length of shortest witness for an x in L.

Proof System Examples

- L = satisfiable boolean formulae
- What's the lowest complexity proof system for this you can come up with?
- What about for L = *un*satisfiable boolean formulae?

Proof System Examples

- For unsatisfiability:
  - Witness = truth table T of Φ
  - P(Φ, T) checks that T is indeed the truth table for Φ, and all entries are zero
- Corresponds to a (failed) brute force search for a solution
- Exponential Complexity
- Is there a proof system for UNSAT with poly complexity? (Does NP = Co-NP?)

Resolution Proof System

- The **Resolution Rule:**

  For clauses B, C and variable x,
  From (B v x) & (C v ¬x) derive (B v C)
- Witness = a sequence of valid derivations starting from the clauses of Φ.
- Sound: (B v x) & (C v ¬x) implies (B v C)
- Complete for unsatisfiability:
  - Every unsatisfiable formula has a derivation of a contradiction (i.e., the empty clause).

Duality

- Truth table proof system gives proofs by failed search for a satisfying assignment.
- Resolution proof system gives proofs by showing the initial clauses (constraints) yield a contradiction. This is a systematic search for additional constraints the solution must satisfy.

High level idea for many solvers

- **Alternate search for solution with search for properties of any solution:**
  - Search for solution in some small part S of the space
  - If search in S fails, search for a reason for this failure, in the form of a new constraint C the solution must satisfy.
  - Search for a solution in a new part of the space, using new constraint to help guide the search
  - Repeat

Notation

- Convenient notational change for SAT:
  - Clauses are sets: (a v ¬b v c) becomes {a, ¬b, c}
  - Formulae become sets of clauses
  - Partial assignments become sets of literals that contain at most one of $\{x_i, \neg x_i\}$ for each i.
  - Assignments contain exactly one of $\{x_i, \neg x_i\}$ for each i.
- Restriction: $\Phi|_{\{x\}}$ is the residual formula under partial assignment {x}, e.g.,

$$\{\{a, \neg b, c\}, \{\neg a, b, d\}\} |_{\{\neg a\}} = \{\{\neg b, c\}\}$$

Basic DPLL (‘60, ‘62)

- Simple tree search for a solution, guided by the clauses of Φ.

```
DPLL-recursive(formula F, partial assignment p)
  (F,p) = Unit-Propagate(F, p);
  If F contains clause {} then
        return (UNSAT, null);
  If F = {} then
        return (SAT, p);
  x = literal such that x and ¬x are not in p;
  (status, p’) = DPLL-recursive($F|_{\{x\}}$, pU{x});
  If status == SAT then
        return (SAT, p’);
  Else return
        DPLL-recursive($F|_{\{\neg x\}}$, pU{¬x});
```

If a clause tells you the value of a var, set it appropriately.

Choose a branch. Many heuristics to choose from.

Basic DPLL

If a clause tells you the value of a variable, set it appropriately.

```
Unit-Propagate(formula F, partial assignment p)
    If F has no empty clause then
        While F has a unit clause {x}
            F = $F|_{\{x\}}$;
            p = p U {x};
    return (F,p)
```

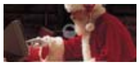
Embellishing DPLL

- Branch Selection Heuristics
- Clause Learning
- Backjumping heuristics
- Watched literals
- Randomized Restarts
- Symmetry breaking
- More powerful proof systems
- …

Branch Selection Heuristics

- Random
- Max occurrence in clauses of min size
- Max occurrence in as yet unsatisfied clauses
- With probability proportional to some function of how often the literal appears in partial assignments that lead to unsatisfiable restricted formulae.
- …

Clause Learning



- When DPLL discovers $F|_p$ is unsatisfiable, it derives (learns) a reason for this in the form of new clauses to add to F.
- What clauses are learned, and how, make **huge** differences in performance.
- Trivially learned clause: if $F|_p$ is unsatisfiable for $p = \{x_1, x_2, \ldots, x_k\}$, derive clause $\{\neg x_1, \neg x_2, \ldots, \neg x_k\}$
  But we want short clauses that constrain the solution space as much as possible…

Clause Learning

- Use the clauses to guide the search:
  - So far we've seen unit-propagation, and search with restriction.
  - We want to learn clauses that let us prune effectively – this requires us to deduce "higher level" reasons why some partial assignment is no good.
  - Use resolution (or some technique) to try to prove that $F|_p$ is unsatisfiable for nodes p high up the tree.

High level: DPLL w/Clause Learning

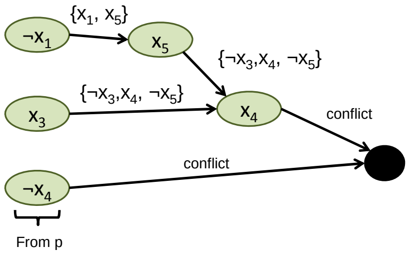
```
DPLL-CL (formula F)
  p = {}
  While(true)
          Choose a literal x  such that x and ¬x are not in p;
          p = pU{x};
          Deduce  status from (F, p);  // SAT, UNSAT, or unknown
          If status == SAT then return (SAT, p);
          If status == UNSAT then
                    Analyze-Conflict(p);     // Add learned clause(s) to F
                    if p = {} then return (UNSAT, null);
                    Else  backtrack; // remove literals from p
                                         // based on learned clause(s)
          If status == unknown then continue;  // branch again
```

Deduce

- Tradeoff between searching more partial assignments (going deeper in the tree), and searching for proofs of unsatisfiability higher up in the tree.
- Currently, deduction is typically just iterated unit-propagation. (Other embellishments to DPLL seem to render more complex deduction unhelpful in practice.)

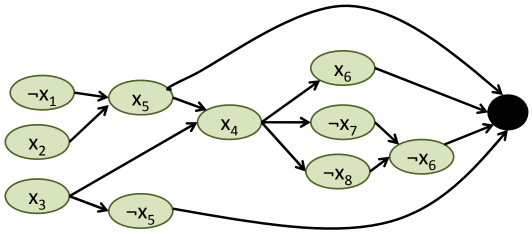
Analyze Conflict: Implication Graph

$p = \{\neg x_1, x_3, \neg x_4\}$
$F = \{\{x_1, x_5\}, \{\neg x_3, x_4, \neg x_5\}\}$



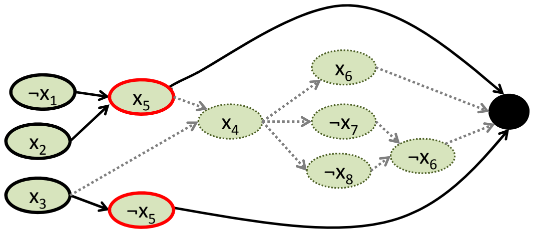
Implication Graph

May contain several sources of conflicts.



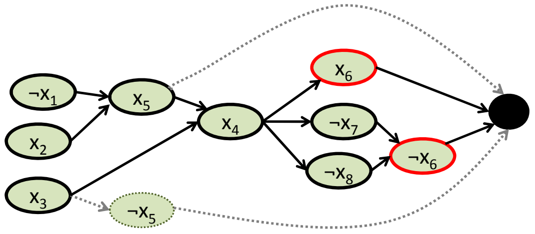
Implication Graph

The conflict graph for conflict variable x_5 (solid edges).



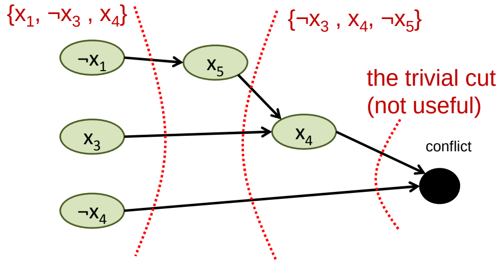
Conflict Graphs

The conflict graph for conflict variable x_6 (solid edges).



Analyze Conflict

Every nontrivial cut of each conflict graph yields a conflict clause. Which one(s) do we add to the clause set?



What Clauses to Learn?

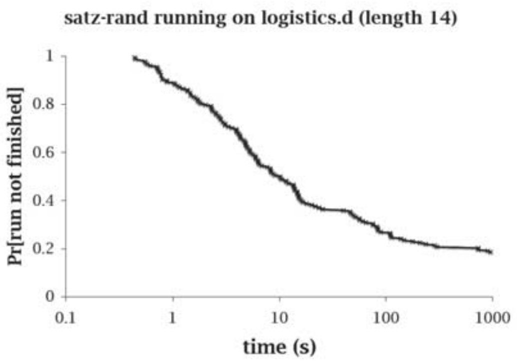
- Can't keep everything -- space is a major bottleneck in practice.
- Various heuristics:
  - First unique implication point
  - First new cut
  - Decision cut
  - …

Backtracking/Backjumping

- For each x in p, maintain:
  - Int depth(x): number of literals in p immediately after x was added to p.
  - Bool flipped(x): did we try the partial assignment with ¬x and all literals at lower depth than x in p?
- If we can derive a conflict clause C containing only x and literals of lower depth in p, then we can backtrack to x:
  - delete all literals with depth > depth(x) from p,
  - If flipped(x) = true, then delete x as well.
- Other heuristics backtrack more aggressively.

Random Restarts

- Restart: keep learned clauses, but throw away p, resample random bits, and start again.
- Essentially a very aggressive backjump.
- Can help performance a lot.
  - Run time distributions appear to be heavy-tailed.



Random Restarts: Heuristics

- Fixed cutoff (always restart after T seconds)
- Cutoff after k restarts is some function f(k)
- Luby et. al. universal restart strategy for f(k)
- f(k) = c*k, c^k, …
- Restart policies based on predictive models of solver behavior:
  - Bayesian approaches, Dynamic Programming,
  - Online submodular function maximization*

Embellishing DPLL

- Branch Selection Heuristics
- Clause Learning
- Backjumping heuristics
- Watched literals
- Randomized Restarts
- Symmetry breaking
- More powerful proof systems
- …

Watched Literals

- Clever lazy data structure

Maintain two literals {x,y} per active clause C that are not set to false. (“C watches x,y”)

```
If x set to true, do nothing
If x set to false,
    For each C watching x,
        either find another variable for C to watch,
        or do unit-propagation on C as appropriate.
    For each previously active C’ containing ¬x,
        set C’ to watch ¬x
If x is unset, do nothing (!)
```

Watched Literals

- Helps quickly find if a clause is satisfied (just look at its watched literals)
- Helps quickly identify clauses ripe for unit-propagation.
- “now a standard method used by most SAT solvers for efficient constraint propagation”
  – Gomes *et. al.* “Satisfiability Solvers”
- Partially explains why deduce step is typically just iterated unit-propagation

Symmetry Breaking

- Symmetry is common in practice (e.g., identical trucks in vehicle routing)
- SAT encoding throws away this info.
- Symmetry is useful for some proofs:
  - e.g., *Pigeon-hole principle*: Impossible to place (n+1) birds into n bins, such that each bin gets at most one bird.

Pigeon Hole Principle

- Exponentially long proofs via resolution.
- Polynomially long proofs via cutting planes

Binary var x(i,j) = assign bird i to bin j.
1) Each bird i gets a bin: $\sum_{(\text{bins } j)} x(i,j) = 1$
2) Each bin j has capacity one: $\sum_{(\text{birds } i)} x(i,j) \leq 1$

Summing (1) over birds: $\sum_{(\text{birds } i)}\sum_{(\text{bins } j)} x(i,j) = n+1$
Summing (2) over bins: $\sum_{(\text{bins } j)}\sum_{(\text{birds } i)} x(i,j) \leq n$
Combine these to get $0 \leq -1$.

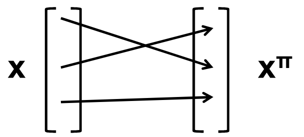
Symmetry Breaking

- Symmetries provided as part of input, or automatically detected (typically via graph isomorphism)
- Impose lexicographically minimal constraints
- Simple case: If $\{x_i : i = 1,2,\ldots, k\}$ are all interchangeable, add constraints $(x_i \vee \neg x_j)$ for all $i < j$.

  "If there's a solution with r of k vars set true, let them be x_1 through x_r"

Symmetry Breaking

- Order the variables, imagine assignment as a vector x.
- Identify permutations π on variables, such that if p is a satisfying assignment, then p•π is.
- Add constraints $x \leq x^\pi$



Example:
$\pi = (2, 1, 3)$

Other Proof Systems

- Truth Tables
- Frege Systems (includes resolution as a special case)
- Extended Resolution: Add new vars
- Resolution w/symmetry detection
- Geometric systems (infer cutting planes)
- …

Incomplete Algorithms

- Returns satisfying assignment or *FAILURE*
- Based on heuristic search for a solution
- Faster than complete algorithms for many classes of satisfiable instances.
- Examples:
  - GSAT, WalkSAT,
  - Survey Propagation/Belief Propagation
  - Local search algs, Simulated Annealing, ILP, …

Greedy-SAT

```
GSAT(formula F)
    For(r = 0 to MAX-ROUNDS)
        Pick random assignment p.
        For (t=0 to MAX-FLIPS)
            If p satisfies F, return p;
            Else
                Find the variable v that if flipped maximizes
                the increase in satisfied clauses of F.
                Flip(v);
    Return FAILURE;
```

```
Flip(boolean v)
    v = ¬v;
```

WalkSAT

```
WalkSAT(formula F)
  For(r = 0 to MAX-ROUNDS)
    Pick random assignment p.
    For (t=0 to MAX-FLIPS)
      If p satisfies F, return p;
      Else
        WalkSAT Local Search Heuristic
  Return FAILURE;
```

WalkSAT Local Search Heuristic

Def: Break count of v relative to p =
number of clauses that flipping v in p renders unsatisfied.

```
Pick unsatisfied clause C at random;
If some v in C has break count = 0, flip v.
Else
    With probability β, flip a random variable in C;
    Else with probability (1-β), flip a variable in C with minimum break count.
```

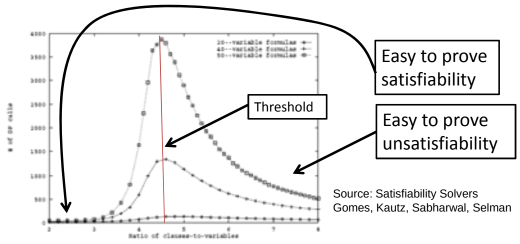
Greedy Move
Random Walk Move

Survey Propagation

- Derived from the cavity method in statistical physics.
- Like DPLL with a special branching heuristic: belief-propagation on objects related to SAT solutions (“covers”)
- Works really well in practice on some random instances – unclear why.

Phase Transitions in SAT

- For random k-SAT instances, time to solve an instance depends on #clauses/#vars

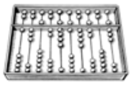


Source: Satisfiability Solvers
Gomes, Kautz, Sabharwal, Selman

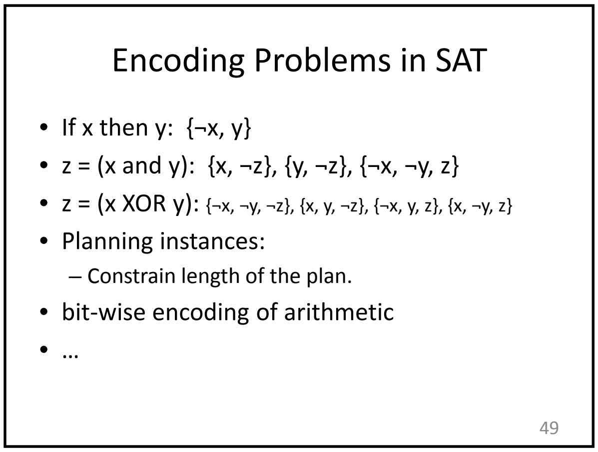
Backdoor Sets

- Given a polynomial time subsolver A and formula F, a set S of variables is a strong backdoor if, whenever the vars in S are fixed by partial assignment p, A solves $F|_p$.
- Some real-world instances of SAT have small backdoor sets (e.g., < 1% of vars).
- Useful in explaining success of certain solvers and restart policies

Model Counting



- Count # of solutions (#P-Complete)
- One idea:
  - Add random parity constraints, until unsatisfiable
  - Each parity constraint eliminates ~1/2 of the solutions.
  - Add k constraints → $\sim 2^{(k-1)}$ solutions



Encoding Problems in SAT

- If x then y: {¬x, y}
- z = (x and y): {x, ¬z}, {y, ¬z}, {¬x, ¬y, z}
- z = (x XOR y): {¬x, ¬y, ¬z}, {x, y, ¬z}, {¬x, y, z}, {x, ¬y, z}
- Planning instances:
  - Constrain length of the plan.
- bit-wise encoding of arithmetic
- …