15-853: Algorithms in the Real World

Suffix Trees

Exact String Matching

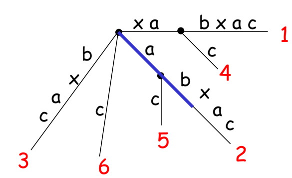
- Given a text T of length m and pattern P of length n
- "Quickly" find an occurrence (or all occurrences) of P in T
- A Naïve solution:
  Compare P with T[i…i+n] for all i --- O(nm) time
- How about O(n+m) time? (Knuth Morris Pratt)
- How about O(m) preprocessing time and O(n) search time?

Suffix Trees

- Preprocess the text in O(m) time and search in O(n) time
- Idea:
  - Construct a tree containing all suffixes of text along the paths from the root to the leaves
  - For search, just follow the appropriate path

Suffix Trees

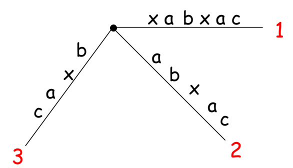
A suffix tree for the string x a b x a c



Search for the string a b x

Constructing Suffix trees

- Naive $O(m^2)$ algo
- For every i, add the suffix S[i .. m] to the current tree



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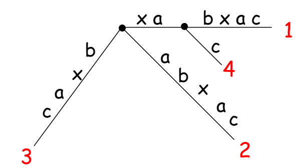

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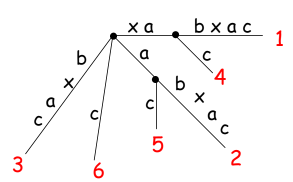


Ukkonen's linear-time algorithm

- We will start with an $O(m^3)$ algorithm and then give a series of improvements
- In stage i, we construct a suffix tree T_i for S[1..i]
- Incrementing T_i to T_{i+1} naively takes $O(i^2)$ time because we insert each of the i suffixes
- Thus a total of $O(m^3)$ time

Going from T_i to T_{i+1}

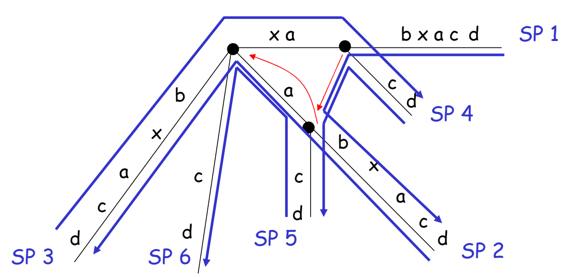
- In the j^{th} substage of stage i+1, we insert S[j..i+1] into T_i. Let S[j..i] = β.
- Three cases
  - Rule 1: The path β ends on a leaf $\Rightarrow$ add S[i+1] to the label of the last edge
  - Rule 2: The path β continues with characters other than S[i+1] $\Rightarrow$ create a new leaf node and split the path labeled β
  - Rule 3: A path labeled β S[i+1] already exists $\Rightarrow$ do nothing.

Idea #1 : Suffix Links

- In each substage, we first search for some string in the tree and then insert a new node/edge/label
- Can we speed up looking for strings in the tree?
- In any substage, we look for a suffix of the strings searched in previous substages
- Idea: Put a pointer from an internal node labeled xα to the node labeled α
- Such a link is called a "Suffix Link"

Idea #1 : Suffix Links

Add the letter d



Suffix Links – Bounding the time

- Steps in each substage
  - Go up 1 link to the nearest internal node
  - Follow a suffix link to the suffix node
  - Follow the path for the remaining string
- First two steps together make up O(m) in each stage
- The third step follows only as many links as the length of the string S[1 .. i]
- Thus the total time per stage is O(m)

Maintaining Suffix Links

- Whenever a node labeled $x\alpha$ is created, in the following substage a node labeled α is created.
  Why?
- When a new node is created, add a suffix link from it to the root, and if required, add a suffix link from its predecessor to it.

Going from $O(m^2)$ to $O(m)$

- Can we even hope to do better than $O(m^2)$?
- Size of the tree itself can be $O(m^2)$
- But notice that there are only 2m edges! – Why?
- Idea: represent labels of edges as intervals
- Can easily modify the entire process to work on intervals

Idea #2 : Getting rid of Rule 3

- Recall Rule 3: A path labeled S[j .. i+1] already exists $\Rightarrow$ do nothing.
- If S[j .. i+1] already exists, then S[j+1 .. i+1] exists too and we will again apply Rule 3 in the next substage
- Whenever we encounter Rule 3, this stage is over – skip to the next stage.

Idea #3 : Fast-forwarding Rules 1 & 2

- Rule 1 applies whenever a path ends in a leaf
- Note that a leaf node always stays a leaf node – the only change is to append the new character to its edge using Rule 1
- An application of Rule 2 in substage j creates a new leaf node
  This node is then accessed using Rule 1 in substage j in all the following stages

Idea #3 : Fast-forwarding Rules 1 & 2

- Fast-forward Rule 1 and 2
  - Whenever Rule 2 creates a node, instead of labeling the last edge with only one character, implicitly label it with the entire remaining suffix
- Each leaf edge is labeled only once!

Loop Structure

i
- j — rule 2 (follow suffix link)
- j+1 — rule 2 (follow suffix link)
- j+2 — rule 3

i+1
- j+2 — rule 3

i+2
- j+2 — rule 3

i+3
- j+2 — rule 2
- j+3 — rule 3

- Rule 2 gets applied once per j
- Rule 3 gets applied once per i

Another Way to Think About It

S

↑ j ↑ i

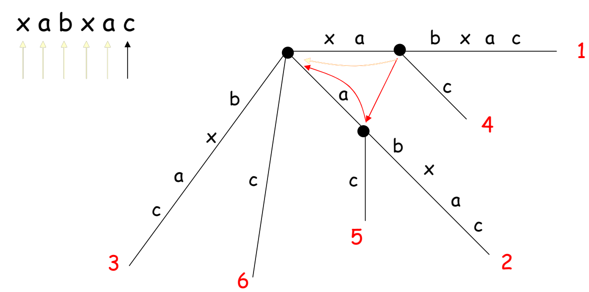
insert finger
increment when S[j..i] not in tree (rule 2)
1) insert S[j..n] into tree by branching at S[j..i-1]
2) create suffix pointer to new node at S[j..i-1] if there is one
3) use parent suffix pointer to move finger to j+1

search finger
increment when S[j..i] in tree (rule 3)

Invariants:
1. j is never after i
2. S[j..i-1] is always in the tree

An example

x a b x a c



Leaf edge labels are updated by using a variable to denote the end of the interval

Complexity Analysis

- Rule 3 is used only once in every stage
- For every j, Rule 1 & 2 are applied only once in the j^{th} substage of all the stages.
- Each application of a rule takes O(1) steps
- Other overheads are O(1) per stage
- Total time is O(m)

Extending to multiple lists

- Suppose we want to match a pattern with a dictionary of k strings
- Concatenate all the strings (interspersed with special characters) and construct a common suffix tree
- Time taken = O(km)
- Unneccesarily complicated tree; needs special characters

Multiple lists – Better algorithm

- First construct a suffix tree on the first string, then insert suffixes of the second string and so on
- Each leaf node should store values corresponding to each string
- O(km) as before

Longest Common Substring

- Find the longest string that is a substring of both S_1 and S_2
- Construct a common suffix tree for both
- Any node that has leaf nodes labeled by S_1 and S_2 in the subtree rooted at it gives a common substring
- The "deepest" such node is the required substring
- Can be found in linear time by a tree traversal.

Common substrings of M strings

- Given M strings of total length n, find for every k, the length l_k of the longest string that is a substring of at least k of the strings
- Construct a common suffix tree
- For every internal node, find the number of distinctly labeled leaves in the subtree rooted at the node
- Report l_k by a single tree traversal
- O(Mn) time – not linear!

Lempel-Ziv compression

- Recall that at each stage, we output a pair (p_i, l_i) where $S[p_i .. p_i+l_i] = S[i .. i+l_i]$
- Find all pairs (p_i,l_i) in linear time
- Construct a suffix tree for S
- Label each internal node with the minimum of labels of all leaves below it – this is the first place in S where it occurs. Call this label c_v.
- For every i, search for the string S[i .. m] stopping just before $c_v \geq i$. This gives us l_i and p_i.