Algorithms in The Real World
Fall 2002
Homework Assignment 2 – Solutions

Problem 1. Suppose that a bipartite graph with n nodes on the left and n nodes on the right is constructed by connecting each node on the left to d randomly-selected nodes on the right (each chosen with probability $1/n$). All random choices are made independently, and there is no restriction on the degree of a node on the right, i.e., nodes on the right may have any degree from 0 to dn. Show that for any fixed $\beta > 1$, and any fixed $d > \beta + 1$, there exists a fixed α such that the graph has (α, β) expansion with probability >0.

Solution. (Maria-Florina Balcan).

We know that, in general, a graph has (α, β) expansion if every subset of $k < \alpha n$ nodes on the left has at least βk neighbors on the right, where α and β are fixed constants, which satisfy $0 \leq \alpha \leq 1$, $\beta > 0$.

Let's suppose first that a (n, n) bipartite graph is created in the following way: for each node on the left, we pick a random subset of d nodes on the right as its neighbors (so each node on the left has exactly d distinct neighbors on the right).

We have $\beta > 1$ fixed, d fixed, $d > \beta + 1$. We choose α such that:

$$\alpha = \left(\frac{\beta^{\beta-d}}{3e^{\beta+d+1}}\right)^{\frac{1}{d-\beta-1}}$$

It is clear from above that $0 \leq \alpha \leq 1$.

We show that for this choice, our graph has (α, β) expansion with probability >0. We have:

$$\begin{gathered} P(\text{graph has } (\alpha,\beta) \text{ expansion property}) = \\ = 1 - P(\text{graph does not have } (\alpha,\beta) \text{ expansion property}) = 1 - p_1 \end{gathered}$$

where:

$$\begin{gathered} p_1 = P(\text{graph does not have } (\alpha,\beta) \text{ expansion property}) = \\ = P(\text{there exist } k \leq \alpha n \text{ nodes on the left that have at most } \beta k \text{ neighbors on} \\ \text{the right}) \end{gathered}$$

Consider $1 \leq k \leq \alpha n$; denote A_k the event that a subset of k vertices on the left has fewer than βk neighbors on the right. It's clear that:

$$p_1 \leq \sum_{1 \leq k \leq \alpha n, k \in N} P(A_k)$$

Fix a subset S of k nodes on the left, and a subset T of βk nodes on the right. There are $\binom{n}{k}$ ways of choosing S and $\binom{n}{\beta k}$ ways of choosing T. The probability that all neighbors of S lie inside T is less or equal to :

$$\frac{\binom{\beta k}{d}^k}{\binom{n}{d}^k}$$

So,

$$P(A_k) \leq \binom{n}{k}\binom{n}{\beta k}\frac{\binom{\beta k}{d}^k}{\binom{n}{d}^k}$$

which is the probability that all dk edges emanating from some k nodes on the left fall within some βk nodes on the right. But then we have:

$$P(A_k) \leq \binom{n}{k}\binom{n}{\beta k}\frac{\binom{\beta k}{d}^k}{\binom{n}{d}^k}$$
$$P(A_k) \leq \left[\left(\frac{ne}{k}\right)^k \left(\frac{ne}{\beta k}\right)^{\beta k} \left(\frac{\beta k e}{d}\right)^{dk}\right] / \left(\frac{n}{d}\right)^{dk}$$
$$P(A_k) \leq \left(n^{\beta+1-d} \cdot k^{d-\beta-1} \cdot \beta^{d-\beta} \cdot e^{\beta+1+d}\right)^k$$

Using $k \leq \alpha n$, we have:

$$P(A_k) \leq \left(\alpha^{d-\beta-1} \cdot \beta^{d-\beta} \cdot e^{\beta+1+d}\right)^k$$

Due to the choice of α, it follows that:

$$P(A_k) \leq \left(\frac{\beta^{\beta-d}}{3e^{\beta+d+1}} \cdot \beta^{d-\beta} \cdot e^{\beta+1+d}\right)^k = \left(\frac{1}{3}\right)^k$$

We have thus showed that:

$$p_1 \leq \sum_{1 \leq k \leq \alpha n, k \in N} P(A_k) \leq \sum_{k \geq 1} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$

So, $p_1 < 1$, and thus the probability that the graph has (α, β) expansion property is $1 - p_1 > 0$.

We have just proved that there exists a fixed α such that the graph has the (α, β) expansion property with probability >0.

Problem 2.

A. Prove that the bisection width (i.e. the number of edges that must be removed to separate a graph into two equal-sized parts, within 1) of the complete graph with n vertices is $(n/2)^2$.

B. Prove that the bisection width of the n-node hypercube is $n/2$. This should be proved from below and above. (Hint: show that the complete graph can be embedded in the hypercube so that vertices map to vertices, edges in the complete graph map to paths in the hypercube, and each hypercube edge ends up supporting the same number of paths.)

Solution. (Ryan Williams).

A. Let G be a complete graph.

We will prove that the bisection width of a complete graph with n nodes is $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. For n even this is exactly $(n/2)^2$, and for n odd this is $\frac{n+1}{2} \cdot \frac{n-1}{2}$, which corresponds to a separating the graph in two "equal-sized parts, within 1".

Clearly, there exists a bisection of G, of size $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, if we place $\lfloor \frac{n}{2} \rfloor$ of G's nodes in a set S, and the other $\lceil \frac{n}{2} \rceil$ nodes in a set T. Each node in S is responsible for exactly $\lceil \frac{n}{2} \rceil$ edges crossing the cut (S, T) and $|S| = \lfloor \frac{n}{2} \rfloor$, so this is a $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ bisection.

Notice the S and T were arbitrary, so this holds for all bisections of G.

B. We can easily think of an $n = 2^k$ node hypercube as a set of strings over $\{0,1\}^k$, with an edge between string x to y iff $h(x, y) = 1$ (where h is the Hamming distance). So let $E = \{(x, y) : h(x, y) = 1\}$.

Consider the bisection (S, T) where $S = \{0b : 0b \in \{0,1\}^k\}$, $T = \{1b : 1b \in \{0,1\}^k\}$. Given any $s \in S$, note there is a unique $t \in T$ such that $(s, t) \in E$, and viceversa, i.e. a 1-1 correspondence between members of S and members of T. There are 2^{k-1} nodes in S, hence $2^{k-1} = n/2$ edges in the bisection. So, the bisection width is at most $n/2$.

Now, for the lower bound. We embed a directed complete graph K_{2^k}, into the hypercube. First, number the nodes in K_{2^k} (from 0 to 2^{k-1}), and associate nodes in K_{2^k} with nodes in the hypercube by associating n with its binary encoding $b(n)$. Then associate each directed edge (u, v) in K_{2^k} with a path in the hypercube from $b(u)$ to $b(v)$. Note a path in the hypercube of length k from $b(u)$ to $b(v)$ can be thought of as a sequence of k bit flips that start with $b(u)$ and end with $b(v)$.

Remember from **A** that the bisection width of K_{2^k} is $(2^k/2)^2 = (n/2)^2$. Let (S, T) be a bisection of the hypercube. By our argument above, there is a corresponding (S', T') bisection of K_{2^k} with labels on the nodes. Consider any edge $(b(u), b(v))$ crossing the bisection (S, T). There are $2^k/2 = n/2$ pairs of nodes $\{b(s), b(t)\}$ in the hypercube, where $b(s) \in S$ and $b(t) \in T$, that take only the edge $(b(u), b(v))$ in the path from $b(s)$ to $b(t)$ (and no other edges crossing the bisection). Therefore, if the bisection width of the hypercube were less than $n/2$, then the number of paths in the hypercube that use those edges in the corresponding (S', T') bisection of K_{2^k} is less than $(2^k/2)^2$, which contradicts the result from **A**.

Problem 3. In class, and in the Karypis and Kumar reading, we covered a multilevel edge-separator algorithm. In this problem you need to generalize this technique to work for vertex separators directly (do not use a postprocessing stage). In particular:

1. Argue why coarsening using a maximal matching is or is not still appropriate.

2. Describe what we should keep track of when coarsening (contracting the graph) (e.g. on the edge separator version each edge kept a weight representing the number of original edges between two multivertices).

3. Describe how we project the solution of the coarsened version back onto the original graph (the recursive solution must return a vertex separator).

4. Describe a variant of Kernighan-Lin or (preferably) the Fiduccia-Mattheyses heuristic for vertex separators. Be explicit about what the gain metric is.

Solution. (Cha Zhang)

1. Coarsening using a maximal matching is still appropriate. Because if at a coarser level G_i we have a vertex separator that has k vertices, then after we expand the graph back to G_{i-1}, we have a vertex separator that has maximum $2k$ vertices. There won't be edges newly generated between the two subsets that are being separated. Hence this provides still a relatively good separator.

2.When coarsening the graph, we need to keep a weight for each node, representing how many nodes were merged to this one.

3. As stated in 1, when projecting the coarsened version back to the original graph, the result is still a reasonably good vertex separator. We just expand the separator and the two other members of the partition separately. The number of edges between A and C, or between C and B may increase, because each end of an edge may expand to two nodes in the next level, and thus for each edge in the coarser graph we may obtain 4 edges in the finer version. However, we only care about the number of vertices, and as already stated in 1, this number (in any of the partition members) can increase at most two times.

4. Variant of the Fiduccia-Mattheyses heuristic:

For each node v, we define gain $G(v)$ as follows: if v belongs to A, then $G(v)$ is the reduction in the size of the vertex separator if we move v into the separator, and then move v' neighbors from the separator into B, if these nodes have no edge to A. Similarly, we can define the gain if v belongs to B. Our algorithm would be:

$FM(G, A, B, C)$
  For every u in A,
    Put u in a priority queue Q_A, based on priority $G(u)$
  For every v in B,
    Put u in a priority queue Q_B, based on priority $G(v)$
  While possible,
    Find the maximum from Q_A and Q_B and do the corresponding
    vertex-moving operation (as in the definition of G).
    Update the gains.

Note: We can also count the reduction in size of the separator, when moving vertex v out of it and into A (or B) and bring its neighbors from B (or A, respectively) into the separator.

Problem 4. Prove that given a class of graphs satisfying an $O(n^{(d-1)/d})$ edge-separator theorem, all members must have bounded degree.

or (more challenging)

Prove that given a class of graphs satisfying an $O(n^{(d-1)/d})$ vertex-separator theorem, the edges of any member can be directed so that the out-degree is bounded. You can use the fact mentioned in the class that such graphs have bounded density (i.e. the average degree is bounded).

Solution. (Doru-Cristian Balcan)

I We have to prove that, given S a class of graphs satisfying a $n^{(d-1)/d}$ edge separator theorem, all the members must have bounded degree. Let's suppose the opposite:

$$\forall t \in N, \exists G_t \in S \text{ such that } deg(G) \geq t \Leftrightarrow$$
$$\Leftrightarrow \forall t \in N, \exists G_t \in S, \exists v_t \in V_t \text{ such that } deg(v_t) \geq t \qquad (*)$$

In order for S to satisfy a $n^{(d-1)/d}$ edge separator theorem, it is necessary (by definition) that S be closed with respect to the subgraph operation. That is, if $G_t \in S$, then all its subgraphs are in S, satisfying the "$n^{(d-1)/d}$ condition".

From (*) it follows that, for every $t \in N$, there exists a graph containing a vertex of degree larger than t. If from G_t we extract the star graph formed by this vertex v_t and its neighbors, it follows that for every $t \in N$, S contains a member (a star graph) with at least $t+1$ nodes.

From the course, we know that the class of the star graphs does not satisfy a $O(n^{(d-1)/d})$, but a $O(n)$ edge separator theorem. But this simply contradicts our supposition.

In conclusion, the degrees of all S's members must be bounded by a positive constant.

II Let's take a class of graphs satisfying a $O(n^{(d-1)/d})$ vertex separator theorem. From the course, we know that the average degree of such graphs is bounded. Let k be such a bound. Then, for a graph G with n vertices, we have:

$$\frac{1}{n}\sum_{i=1}^{n} deg(v_i) \leq k \Leftrightarrow \sum_{i=1}^{n} deg(v_i) \leq nk$$

We study the possibility of giving an orientation for the edges of G, such that the out-degree of all vertices is bounded (basically, we mean that the maximum out-degree bound does not depend on n).

For a directed graph, we know that:

$$\sum_{i=1}^{n} deg^-(v_i) = \sum_{i=1}^{n} deg^+(v_i) = |E|, \text{ and } \sum_{i=1}^{n} deg^-(v_i) + \sum_{i=1}^{n} deg^+(v_i) = \sum_{i=1}^{n} deg(v_i)$$

It follows that $\sum_{i=1}^{n} deg^{+}(v_i) = \frac{1}{2}\sum_{i=1}^{n} deg(v_i) \leq \frac{k}{2}n$ (so, we might have an idea of this bound's magnitude).

We will prove the out-degree boundedness by induction over the graph size. We use mainly the fact that the class of graphs considered here is closed with respect to the subgraph operation.

Let v_1 be the vertex with the smallest degree. We shall orient its incident edges "outwards". Obviously, then, $deg^{+}(v_1) = deg(v_1)$.

Since $\sum_{i=1}^{n} deg(v_i) \leq kn$ and $deg(v_1) = \min_{v \in V} deg(v)$, it follows that $deg(v_1) \leq k$.

The problem would be solved if we found an orientation of all the other edges of the graph, such that the out-degree of all the vertices (different from v_1!) is bounded.

But we know this, from the induction hypothesis: by taking out v_1 and its incident edges, we do not interfere with other vertices' future out-degree, and we basically reduce the problem to an identical one, but for a smaller graph size. So, we can give an orientation of the remaining edges (those non-incident with v_1) such that the out-degree of the vertices different from v_1 is bounded by a constant b_0.

If we choose $b = \max(k, b_0)$, then it follows that we can orient the edges of any graph in the class, such that the maximum out-degree is smaller than a constant b.

With this, the problem is solved completely.

Problem 5. Consider applying divide-and-conquer to graphs and let's say that merging two recursive solutions take $f(s)$ time, where s is the number of edges separating the two graphs. For each of the following $f(s)$, and assuming you are given an edge-separator tree for which all separators for the subgraphs of size n are $1/2 - 2/3$ balanced and bounded by $kn^{1/2}$, what is the running time of such an approach.

1. s
2. $s \log s$
3. s^2
4. s^4

Solution. (Hubert Chan)

Let $T(n)$ be the running time in the case of a graph of size n. The recursive formula is:

$$\begin{array}{ll} T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + f(k\sqrt{n}), & \text{for } n > n_0 \\ T(n) = O(1), & \text{for } n \leq n_0 \end{array}$$

where n_0 is the threshold to stop the recursion.

1. $f(s) = s$

Assume for all $n \leq n_0$, we have $A > 0$ large enough such that:

$$T(n) \leq An - Bk\sqrt{(n)} \text{ for } n > n_0, \text{ where } B = \frac{1}{\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}} - 1}.$$

The inductive step:

$$
\begin{aligned}
T(n) &= T(\tfrac{n}{3}) + T(\tfrac{2n}{3}) + f(k\sqrt{n}) \\
&\leq \tfrac{1}{3}An - Bk\sqrt{\tfrac{n}{3}} + \tfrac{2n}{3}A - Bk\sqrt{\tfrac{2n}{3}} + k\sqrt{n} \\
&= An - k\sqrt{n}\left(\frac{\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{3}}}{\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{3}}-1} - 1\right) \\
&= An - Bk\sqrt{n}
\end{aligned}
$$

2. $f(s) = s\log(s)$ (log is base 2)
Using a similar approach, we show:

$$T(n) \leq An - B\sqrt{n}\log n - C\sqrt{n} \text{ for all } n,$$

for some constants A, B, C.

We first figure out what B and C can be, by looking at the inductive step. Denote $P(n) = An - B\sqrt{n}\log n - C\sqrt{n}$. We need to have:

$$
\begin{aligned}
T(n) &= T(\tfrac{n}{3}) + T(\tfrac{2n}{3}) + k\sqrt{n}\log(k\sqrt{n}) \\
&\leq P(\tfrac{n}{3}) + P(\tfrac{2n}{3}) + \tfrac{1}{2}k\sqrt{n}\log n + k(\log k)\sqrt{n}
\end{aligned}
$$

So, we need:

$$P(\tfrac{n}{3}) + P(\tfrac{2n}{3}) + \tfrac{1}{2}k\sqrt{n}\log n + k\log k\sqrt{n} \leq P(n). \ (1)$$

We can easily see that:

(2): $P(\frac{n}{3}) + P(\frac{2n}{3}) + \frac{1}{2}k\sqrt{n}\log n + k\log k\sqrt{n}$
$= A\frac{n}{3} - B\sqrt{\frac{n}{3}}\log\frac{n}{3} - C\sqrt{\frac{n}{3}} + A\frac{2n}{3} - B\sqrt{\frac{2n}{3}}\log\frac{2n}{3} - C\sqrt{\frac{2n}{3}} + \frac{1}{2}k\sqrt{n}\log n +$
$k\log k\sqrt{n}$
$= An - \left[B\frac{1}{\sqrt{3}}\sqrt{n}\log n - B\frac{1}{\sqrt{3}}\sqrt{n}\log 3\right] - \left[B\sqrt{\frac{2}{3}}\sqrt{n}\log n - B\sqrt{\frac{2}{3}}\sqrt{n}\log\frac{3}{2}\right] -$
$C\sqrt{\frac{n}{3}} - C\sqrt{\frac{2n}{3}} + \frac{1}{2}k\sqrt{n}\log n + k\log k\sqrt{n}$
$= An - \sqrt{n}\log n\left[B\frac{1}{\sqrt{3}} + B\sqrt{\frac{2}{3}} - \frac{1}{2}k\right] - \sqrt{n}\left[-B\frac{1}{\sqrt{3}}\log 3 - B\sqrt{\frac{2}{3}}\log\frac{3}{2} + C\sqrt{\frac{1}{3}} + C\sqrt{\frac{2}{3}} - k\log k\right]$

It is quite easy to see that the last member of the above equation is lower or equal to $P(n)$, if we choose:

$B = \max\left\{1, \frac{k}{2\left(1/\sqrt{3}+\sqrt{2/3}-1\right)}\right\}$

$C = \max\left\{1, \frac{k\log k + B\frac{1}{\sqrt{3}}\log 3 + B\sqrt{\frac{2}{3}}\log\frac{3}{2}}{1/\sqrt{3}+\sqrt{2/3}-1}\right\}$

$A = \max\{C+1, 2\}$

3. $f(s) = s^2$

Suppose $T(n) \leq \frac{k^2}{H(\frac{1}{3})}n\log_2 n + An$ for large enough A, such that the inequality holds for $n \leq n_0$ and $H(p) = -p\log_2 p - (1-p)\log_2(1-p)$.

We write $H = H(\frac{1}{3})$ for the rest of the solution. Inductive step:

$$\begin{aligned}
T(n) &= T(\tfrac{n}{3}) + T(\tfrac{2n}{3}) + k^2 n \\
&\leq \tfrac{k^2}{H}\tfrac{n}{3}\log\left(\tfrac{n}{3}\right) + \tfrac{k^2}{H}\tfrac{2n}{3}\log\left(\tfrac{2n}{3}\right) + k^2 n + \tfrac{An}{3} + \tfrac{2An}{3} \\
&= \tfrac{k^2}{H}\left(\tfrac{n}{3}\log n + \tfrac{2n}{3}\log n\right) + \tfrac{k^2}{H} n\left(\tfrac{1}{3}\log\tfrac{1}{3} + \tfrac{2}{3}\log\tfrac{2}{3}\right) + k^2 n + An \\
&= \tfrac{k^2}{H} n\log n + An
\end{aligned}$$

4. $f(s) = s^4$

Again, let $A > 0$ be large enough, so that for $n \geq n_0$, we have:

$T(n) \leq \frac{9}{4}k^4 n^2 + An$
Inductive step:

$$\begin{aligned}
T(n) &= T(\tfrac{n}{3}) + T(\tfrac{2n}{3}) + k^4 n^2 \\
&\leq \tfrac{9}{4}k^4\left(\tfrac{n}{3}\right)^2 + \tfrac{9}{4}k^4\left(\tfrac{2n}{3}\right)^2 + k^4 n^2 + A\left(\tfrac{n}{3}\right) + A\left(\tfrac{2n}{3}\right) \\
&= \tfrac{9}{4}k^4 n^2 + An.
\end{aligned}$$