1. Johnson-Lindenstrauss lemma

Projecting \( n \) points in a \( d \)-dimensional space into \( K \geq 2\varepsilon^{-2}\ln n \) dimensions. There is a mapping \( f : \mathbb{R}^d \to \mathbb{R}^k \) such that \( \forall u, v; (1 - \varepsilon)||u - v||^2 \leq ||f(u) - f(v)||^2 \leq (1 + \varepsilon)||u - v||^2 \)

Idea:
(a) project onto random hyperplane, measure distance, and multiply by \( \frac{d}{K} \)
(b) Show that squared length of random vector is sharply concentrated around the mean, not distorted by more than \( (1 \pm \varepsilon) \) with probability \( \frac{1}{n^2} \)

\[
pr[Y \geq t] \leq \frac{E[Y]}{t} \quad \text{Markov Inequality}
\]
\[
pr[X - \mu_x \geq t\sigma_x] \leq \frac{1}{e^t}, \sigma_x = \sqrt{E[(x - \mu_x)^2]}, \quad \text{ChebyChev Inequality}
\]
\[
E[L_k^2] = \frac{k}{d}, x_1^2 + x_2^2 + \ldots + x_d^2 = 1, \text{ choosing } k \text{ from } d \text{ (view this process as the flipping coins)}.
\]

2. Chernoff Bounds

\( x_1, x_2, ..., x_n \) independent Poisson trials \((0 \text{ or } 1 \text{ outcome, each with probability } p \text{ of being } 1)\).
\( X = \sum_{i=1}^n x_i, \mu_x = np \)

\[
pr[X > (1 + \sigma)\mu] < \left( \frac{e^\sigma}{(1 + \sigma)^n} \right)^n \quad pr[X < (1 - \sigma)\mu] < e^{-n\sigma^2/2} \quad \text{i.e., } n = 1000, p = 0.5, \sigma = 0.1
\]

3. proof of Chernoff bounds

Consider \( E[e^{tx}] \), the moment generating function = \( 1 + tE[x] + \frac{t^2}{2}e[x^2] + \frac{t^3}{6}e[x^3] \)
\( e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \ldots + \frac{t^nx^n}{n!} \)

\[
E[e^{tx}] = E[e^{t \sum_{i=1}^n x_i}] = E[\prod_{i=1}^n e^{tx_i}] = \prod_{i=1}^n E[e^{tx_i}] = \prod_{i=1}^n (pe^t + (1 - p)) = \prod_{i=1}^n (1 + p(e^t - 1)) = (1 + p(e^t - 1))^n < e^{p(e^t - 1)n} = e^{\mu(e^t - 1)}
\]

\[
pr[x > (1 + \sigma)\mu] = pr[e^{tx} > e^{t(1 + \sigma)}\mu] < \frac{E[e^{tx}]}{e^{t(1 + \sigma)}\mu} = \left( \frac{e^{t(e^t - 1)}}{e^{t(1 + \sigma)}} \right)^\mu
\]