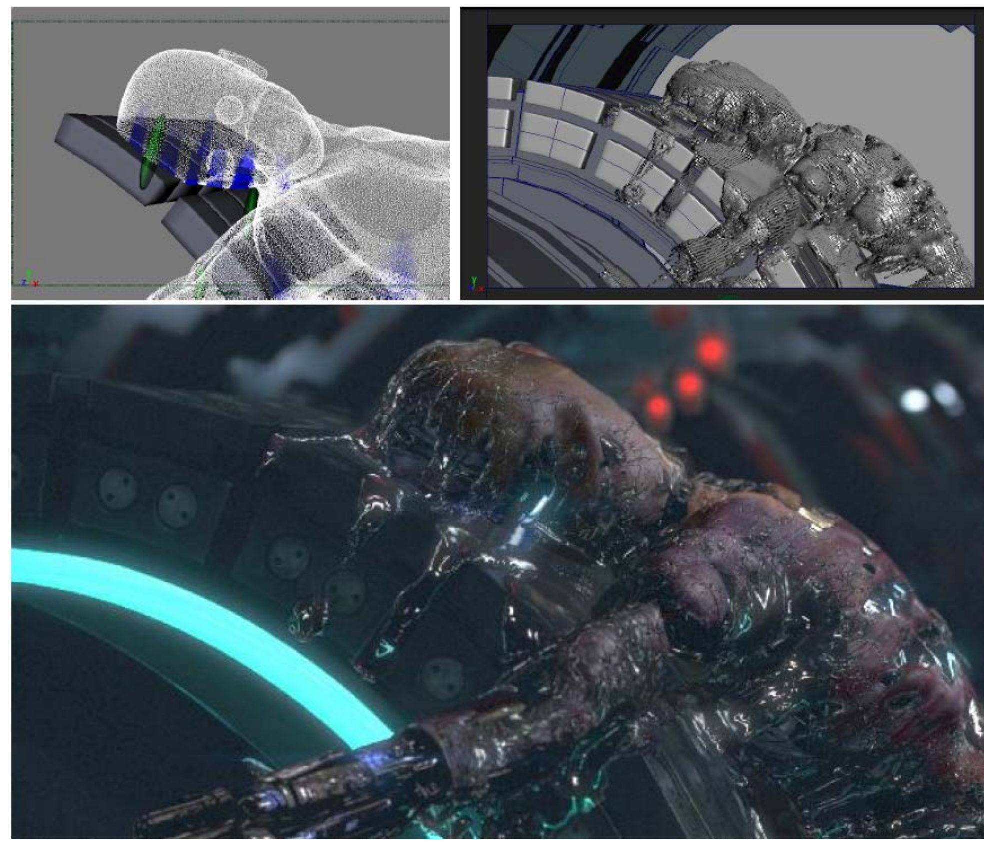
Modeling, Simulating and Rendering Fluids

Thanks to Ron Fediw et al, Jos Stam, Henrik Jensen, Ryan

Applications

- Mostly Hollywood
  - Shrek
  - Antz
  - Terminator 3
  - Many others…



- Games
- Engineering

Animating Fluids is Hard…

- Too complex to animate by hand
  - Surface is changing very quickly
  - Lots of small details

- Need automatic simulations

Ad-Hoc Methods

- Some simple algorithms exist for special cases
  - Mostly waves
- What about water glass?
- Too much work to come up with empirical algorithms for each case

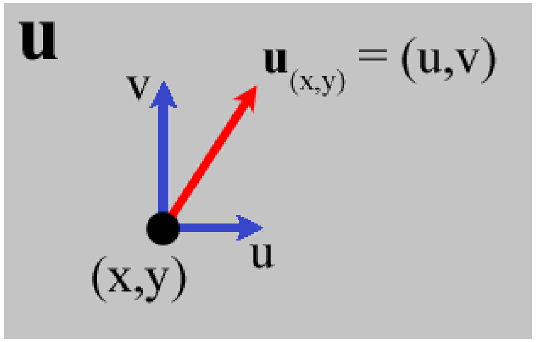
Physically-Based Approach

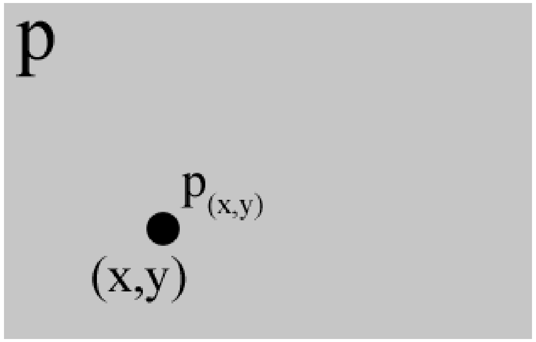
- Borrow techniques from *Fluid Dynamics*
  - Long history. Goes back to Newton…
  - Equations that describe fluid motion

- Use numerical methods to approximate fluid equations, simulating fluid motion
  - Like mass-spring systems

What do we mean by 'Fluid'?

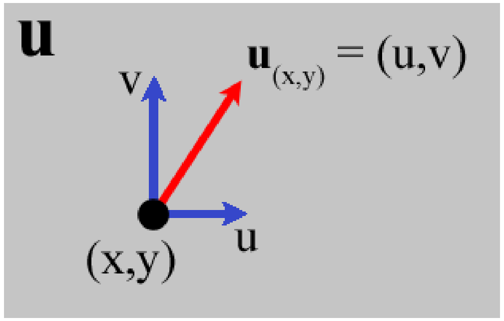
- liquids or gases
- Mathematically:
  - A vector field **u** (represents the fluid *velocity*)
  - A scalar field p (represents the fluid *pressure*)
  - fluid density (d) and fluid viscosity (*v*)



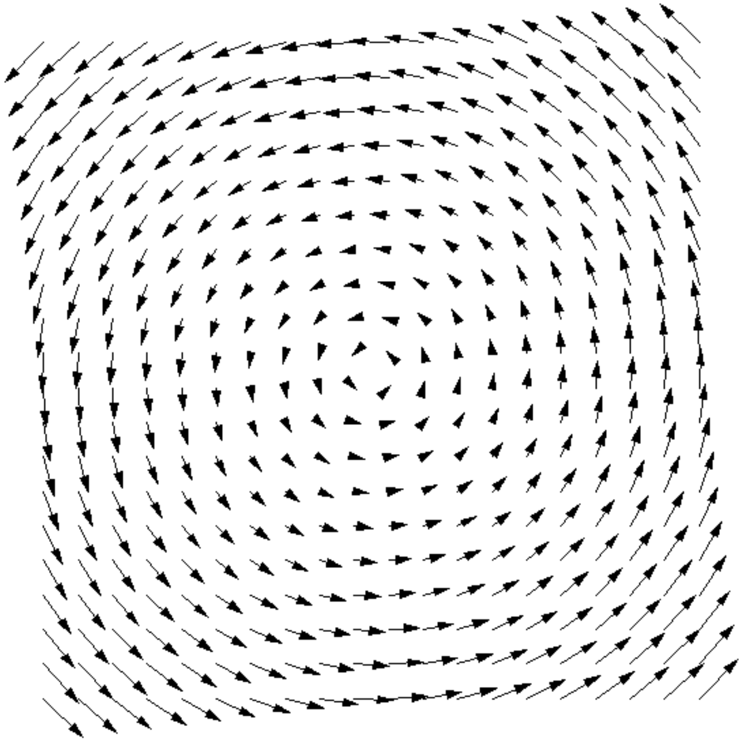


Vector Fields

- 2D Scalar function:
  - f(x,y) = z
  - z is a *scalar* value



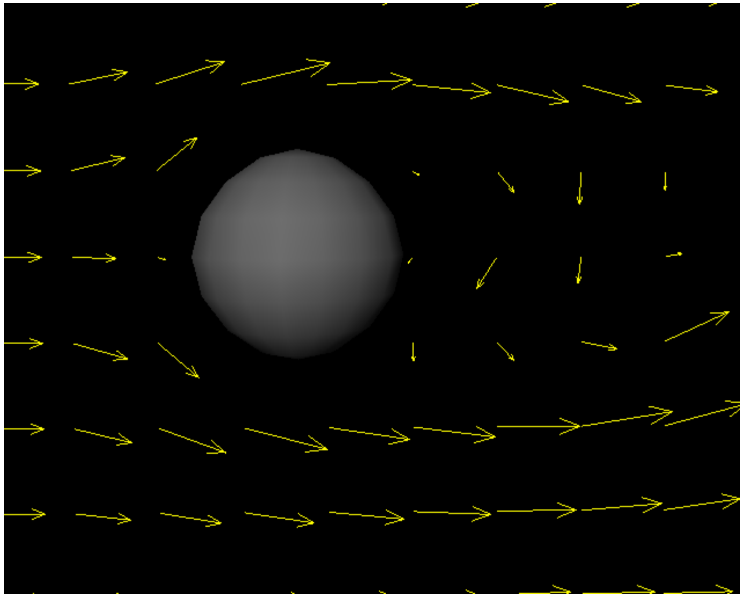
- 2D *Vector* function:
  - **u**(x,y) = **v**
  - **v** is a *vector* value
    - **v** = (x′, y′)

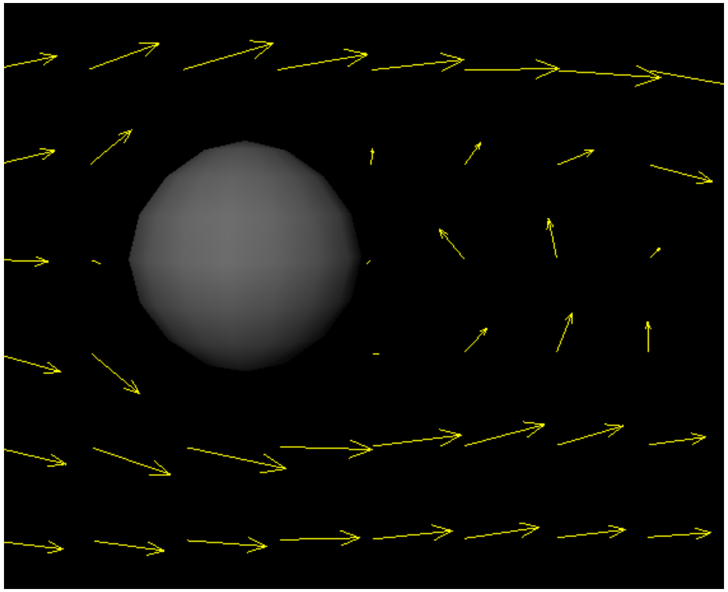


- The set of values **u**(x,y) = **v** is called a *vector field*

Fluid Velocity == Vector Field

- Can model a fluid as a vector field **u**(x,y)
  - **u** is the *velocity* of the fluid at (x,y)
  - Velocity is different at each point in fluid!
- Need to compute *change in vector field*





Particles carry Velocities

- Particle Simulation:
  - Track particle *positions* $\mathbf{x} = (x,y)$
  - Numerically Integrate: *change in position*

$$\frac{d\mathbf{x}}{dt}$$

- Fluid Simulation :
  - Track fluid *velocities* $\mathbf{u} = (u,v)$ at *all points* $\mathbf{x}$ in some fluid volume D
  - Numerically Integrate: *change in velocity*

$$\frac{d\mathbf{u}}{dt}$$

Some Math

- Del Operator:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

- Laplacian Operator:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Gradient:

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right)$$

More Math

- Vector Gradient:

$$\nabla \mathbf{u} = (\nabla u, \nabla v, \nabla w)$$

- Divergence:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Directional Derivative:

$$\mathbf{u} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Navier-Stokes Fluid Dynamics

- Velocity field **u**, Pressure field *p*
  - Viscosity *v*, density *d* (constants)
  - External force **f**

- Navier-Stokes Equation:

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + v\nabla^2\mathbf{u} - \frac{1}{d}\nabla p + \mathbf{f}$$

- Mass Conservation Condition: $\nabla \cdot \mathbf{u} = 0$

Navier-Stokes Equation

- Derived from momentum conservation condition
- 4 Components:
  - Advection/Convection
  - Diffusion *(damping)*
  - Pressure
  - External force *(gravity, etc)*

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right)\mathbf{u} + v\nabla^2\mathbf{u} - \frac{1}{d}\nabla p + \mathbf{f}$$

Mass Conservation Condition

- Velocity field u has zero divergence
  - Net mass change of any sub-region is 0
  - Flow in == flow out
  - Incompressible fluid
- Comes from continuum assumption

$$\nabla \cdot \mathbf{u} = 0$$

Change in Velocity

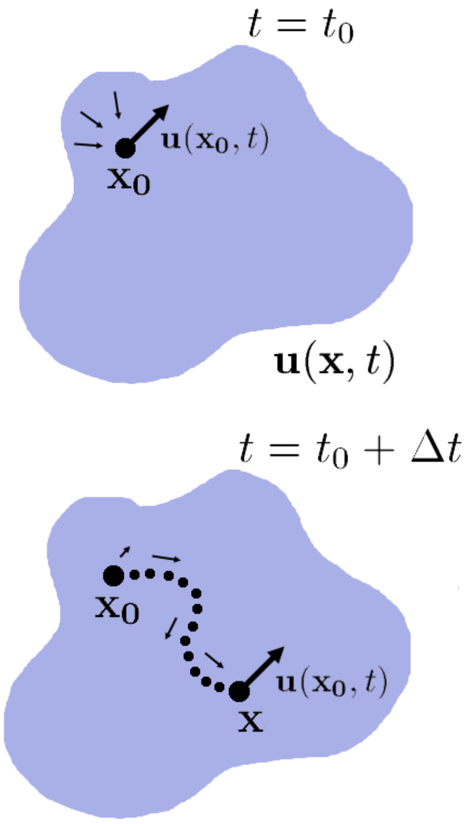
$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right)\mathbf{u} + \nabla \cdot \left(v \nabla \mathbf{u}\right) - \frac{1}{d} \nabla p + \mathbf{f}$$

- Derivative of velocity with respect to time
- *Change* in velocity, or acceleration
  - So this equation models acceleration of fluids

Advection Term

Change in Velocity $= -(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \cdot (v \nabla \mathbf{u}) - \frac{1}{d}\nabla p + \mathbf{f}$

- Advection term
  - Force exerted on a particle of fluid by the other particles of fluid surrounding it
  - How the fluid "pushes itself around"



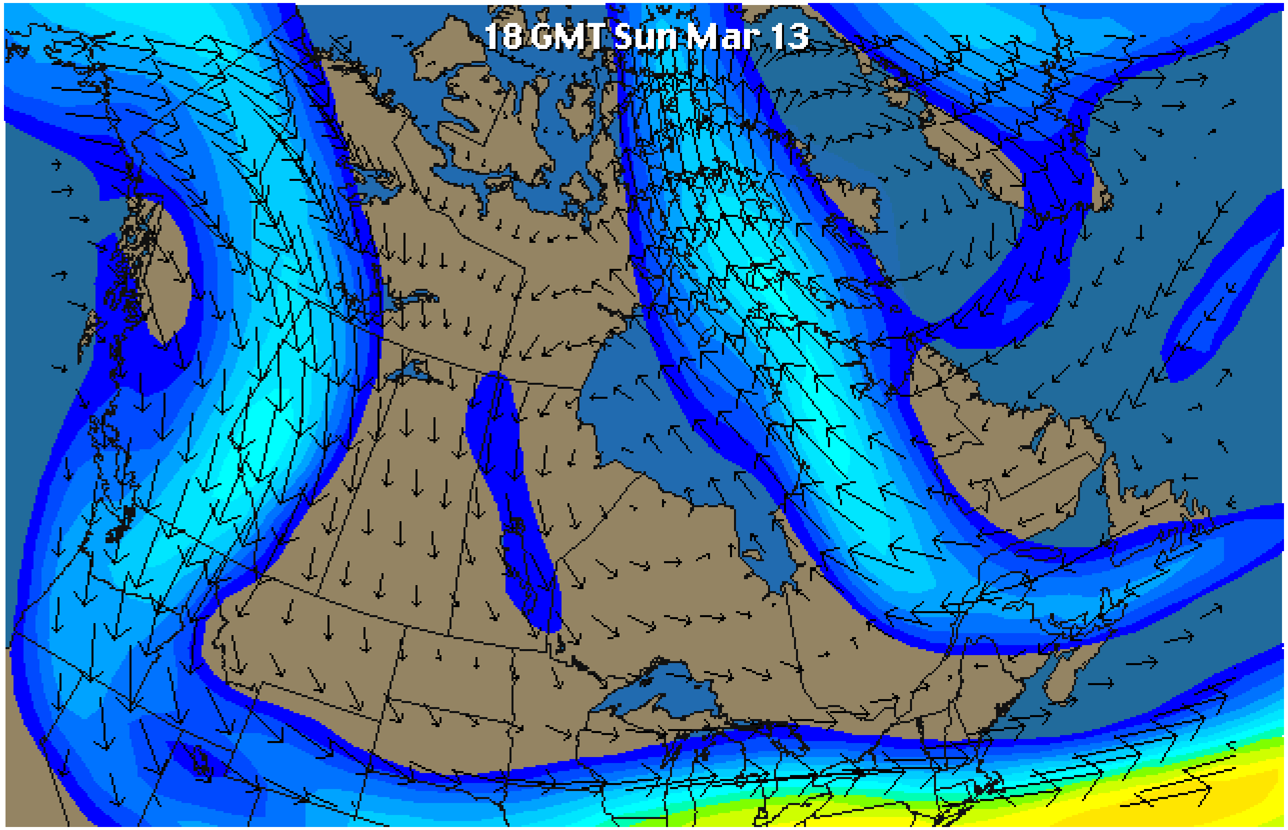
Diffusion Term

$$\text{Change in Velocity} = \textbf{Advection} + \nabla \cdot (v \nabla \mathbf{u}) - \frac{1}{d} \nabla p + \mathbf{f}$$

- Viscosity constant v controls velocity diffusion
- Essentially, this term describes how fluid motion is damped
- Highly viscous fluids stick together
  - Like maple syrup
- Low-viscosity fluids flow freely
  - Gases have low viscosity

Weather: Advection & Diffusion

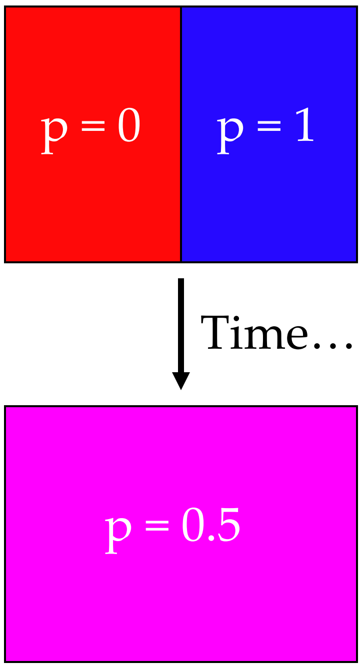
- "Jet-Stream"



Pressure Term

$$\text{Change in Velocity} = \textbf{Advection} + \textbf{Diffusion} - \frac{1}{d}\nabla p + \mathbf{f}$$

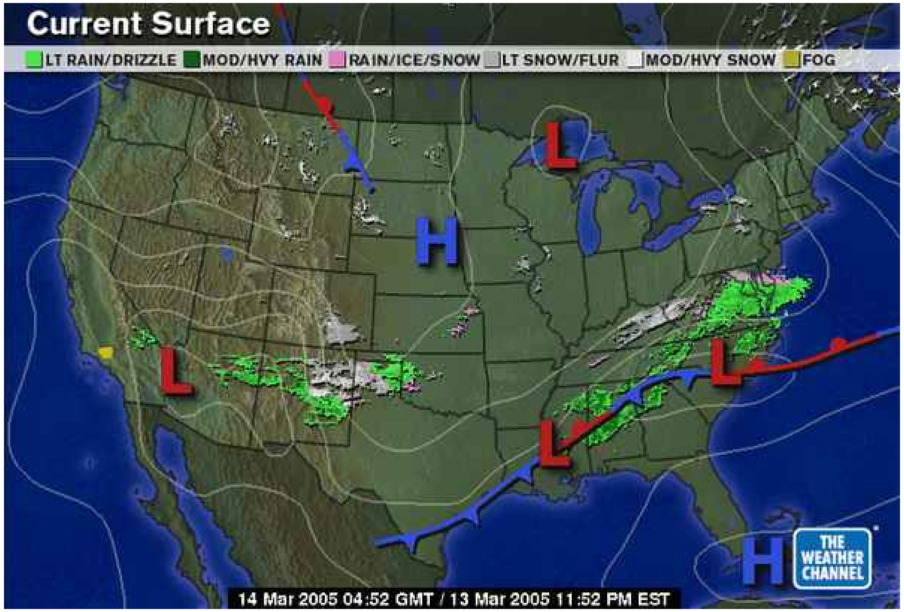
- Pressure follows a diffusion process
  - Fluid moves from high-pressure areas to low-pressure areas



- Moving == *velocity*
  - So fluid moves in direction of largest change in pressure
  - This direction is the *gradient*

Weather: Pressure

- "Fronts" are the boundaries between regions of air with different pressure…
- "High Pressure Zones" will diffuse into "Low Pressure Zones"



Body Force

$$\textbf{Change in Velocity} = \textbf{Advection} + \textbf{Diffusion} - \textbf{Pressure} + \mathbf{f}$$

- Body force term represents external forces that act on the fluid
  - Gravity
  - Wind
  - Etc…

Summary

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} + \nabla\cdot\left(v\nabla\mathbf{u}\right) - \frac{1}{d}\nabla p + \mathbf{f}$$

Change in Velocity = **Advection** + **Diffusion** − **Pressure** + **f**

- Add mass conservation (1 liter in == 1 liter out) constraint:

$$\nabla\cdot\mathbf{u} = 0$$

- Need to simulate these equations…

Incompressible Euler Equations

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{f}$$

self-advection forces

$$\nabla \cdot \mathbf{u} = 0$$

incompressible

(Navier-Stokes without viscosity)

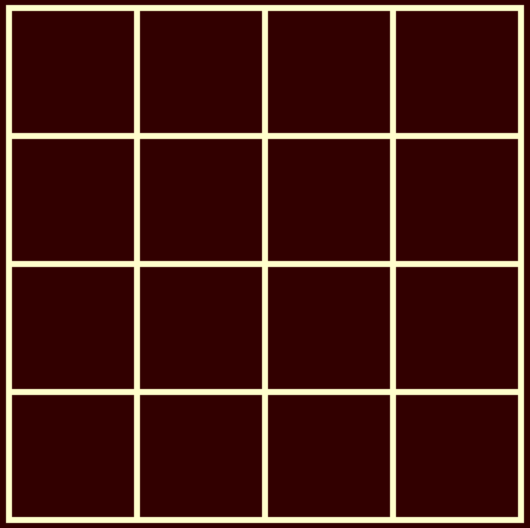
Additional Equations

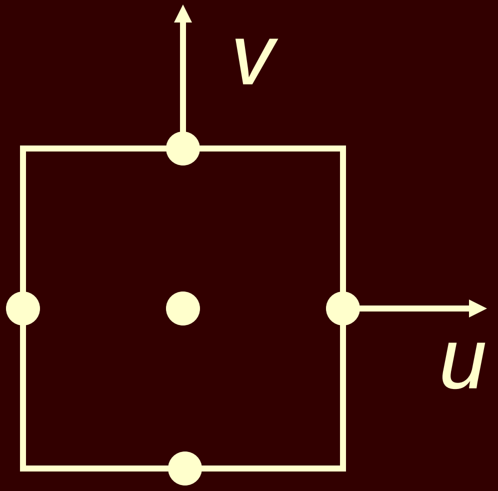
smoke's density
$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho + S$$

temperature
$$\frac{\partial T}{\partial t} = -(\mathbf{u} \cdot \nabla)T + H$$

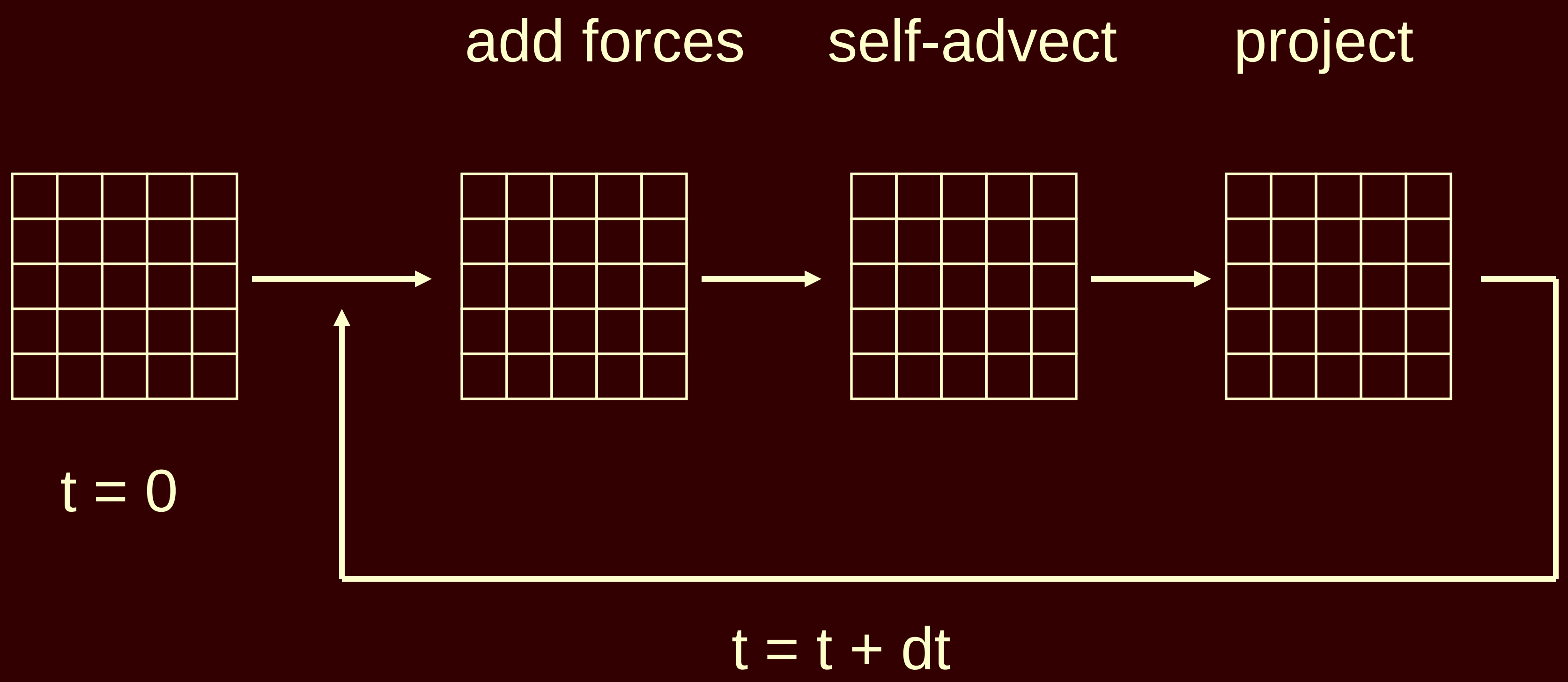
$$\mathbf{f} = -\alpha \, \rho \, \mathbf{z} + \beta \, (T - T_{\mathrm{amb}}) \, \mathbf{z}$$

Discretization





Algorithm



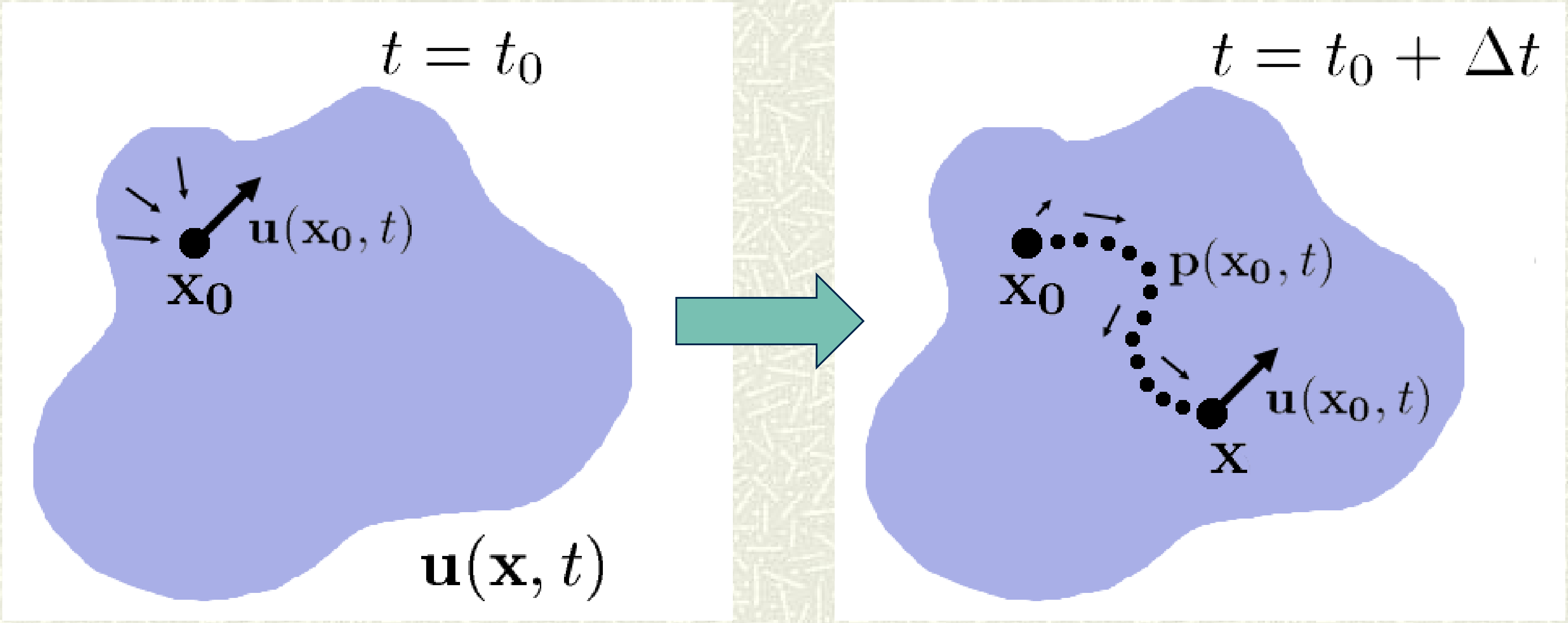
Step 1 – Add Force

- Assume change in force is small during timestep
- Just do a basic forward-Euler step

$$\mathbf{w}_1(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) + \Delta t\,\mathbf{f}(\mathbf{x}, t)$$

- *Note:* **f** *is actually an acceleration?*

Step 2 - Advection

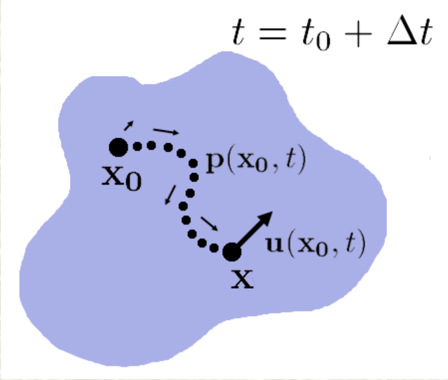


Method of Characteristics

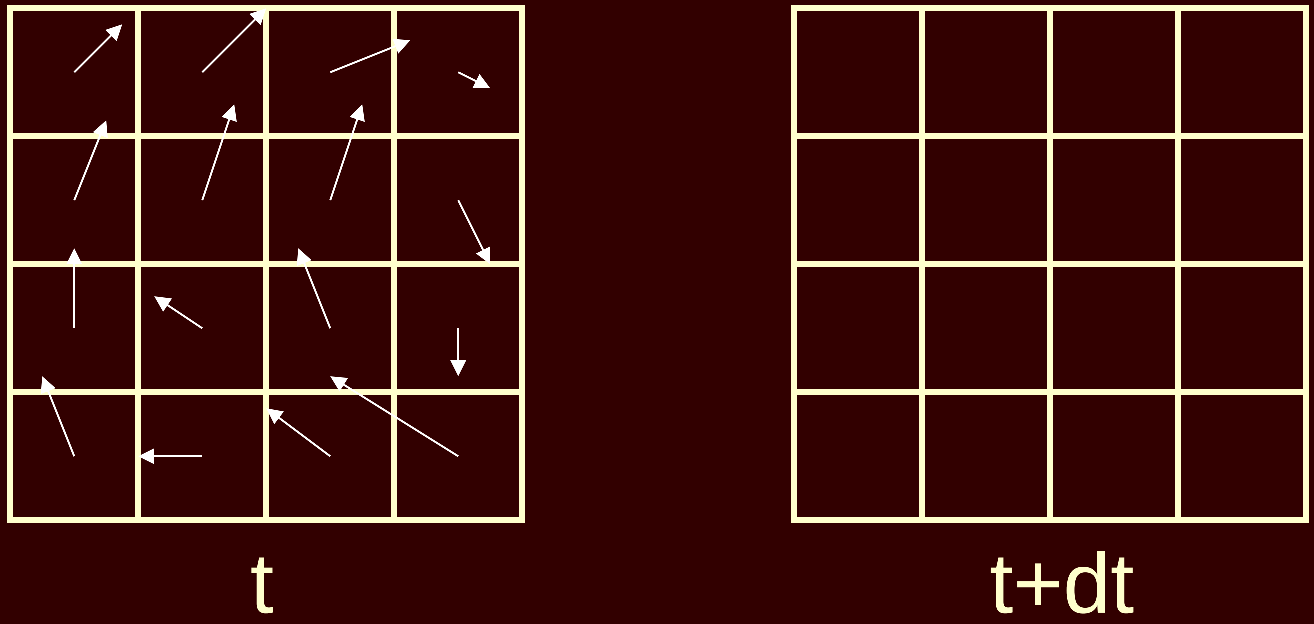
- **p** is called the *characteristic*
  - Partial streamline of velocity field **u**
  - Can show **u** does not vary along streamline
- Determine **p** by tracing backwards

$$\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$$

- Unconditionally stable
  - Maximum value of $\mathbf{w}_2$ is never greater than maximum value of $\mathbf{w}_1$

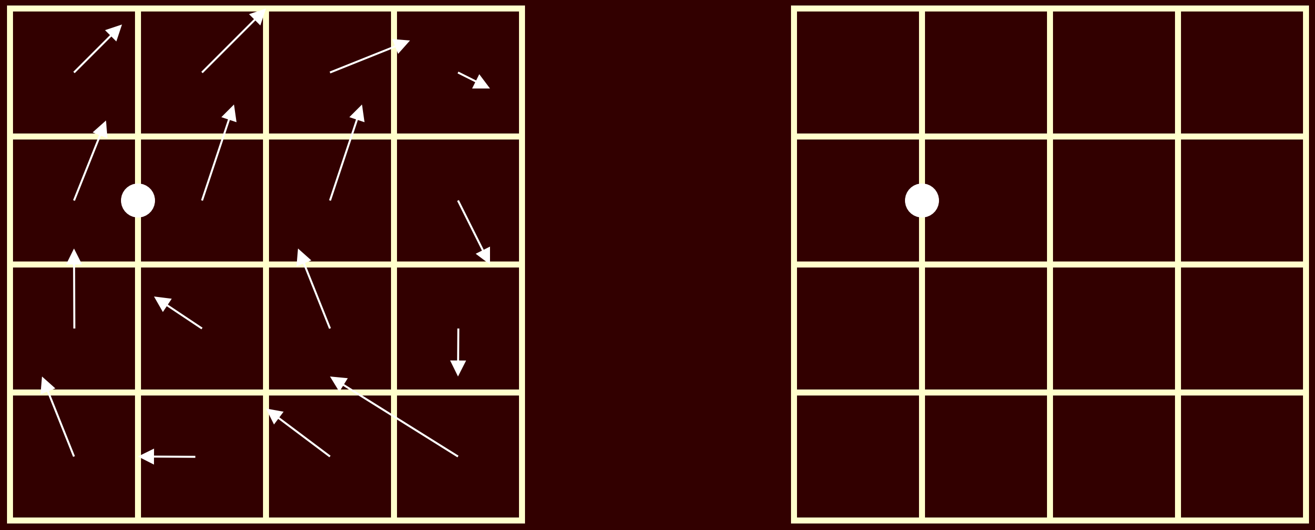


Self-Advection



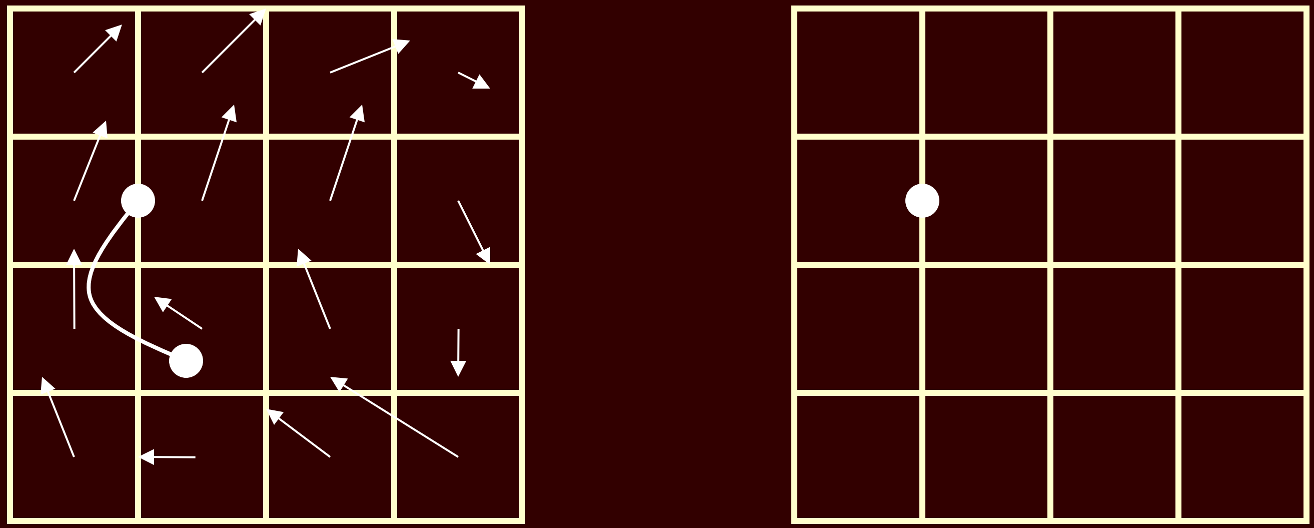
Semi-Lagrangian solver (Courant, Issacson & Rees 1952)

Self-Advection



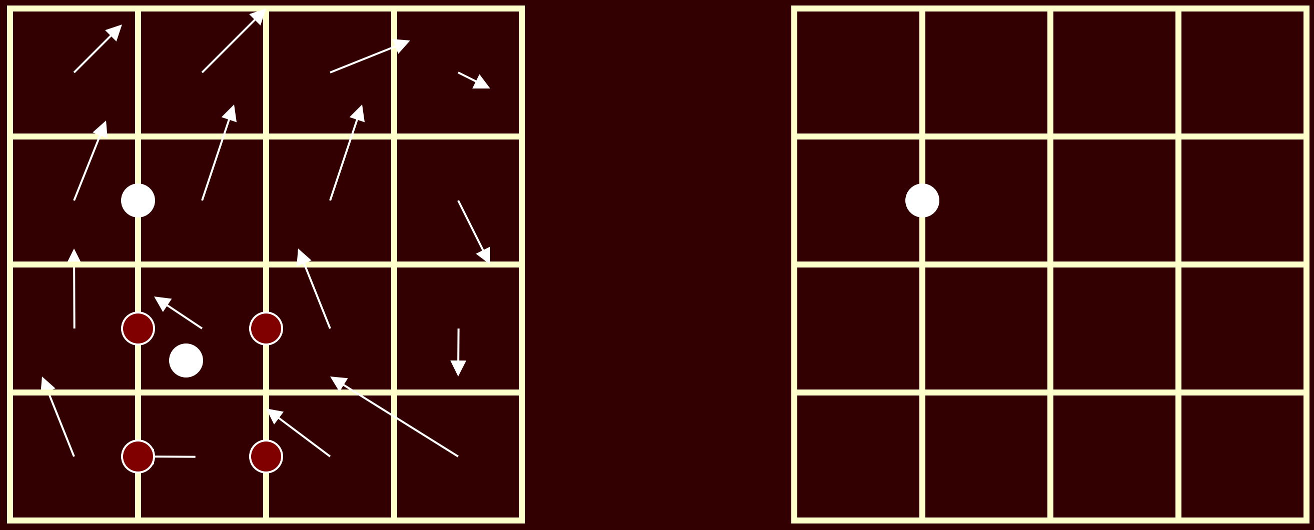
For each u-component…

Self-Advection



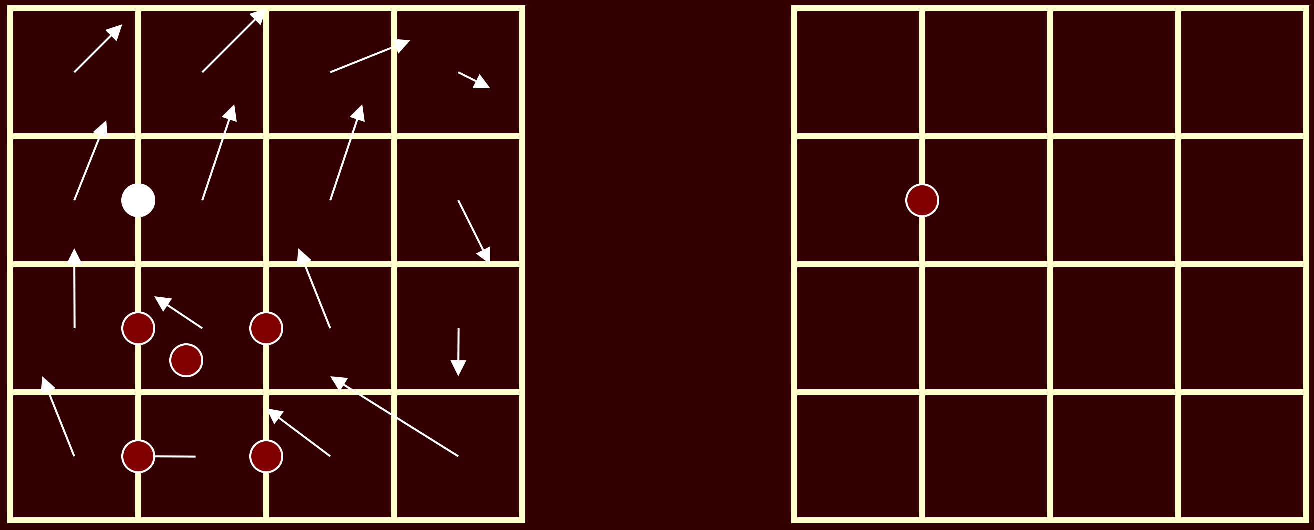
Trace backward through the field

Self-Advection



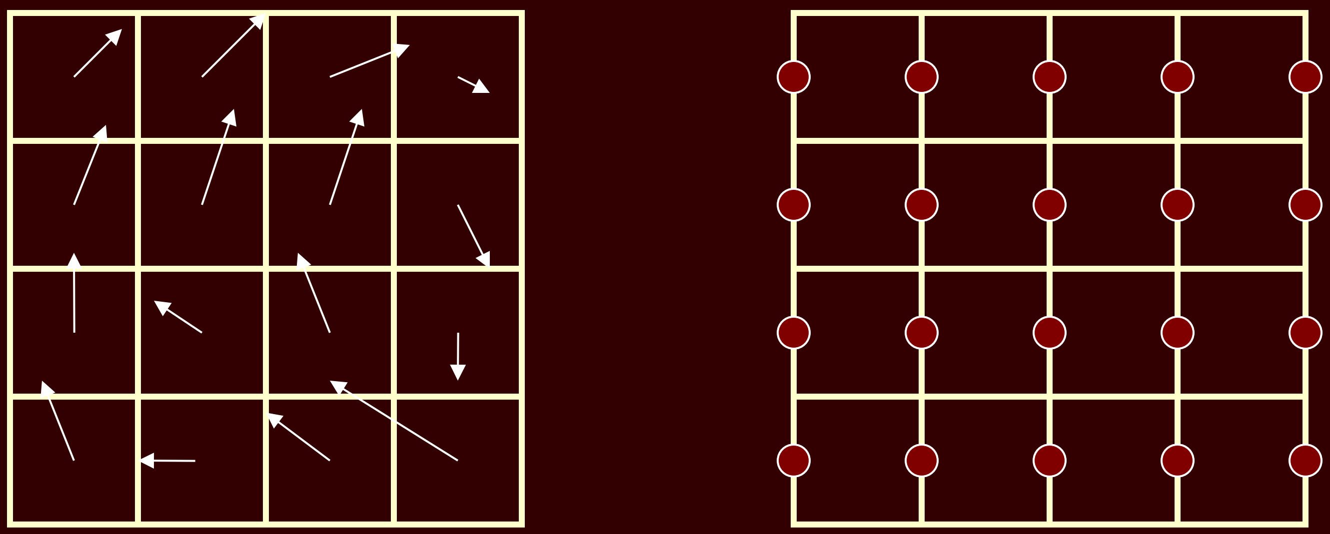
Interpolate from neighbors

Self-Advection



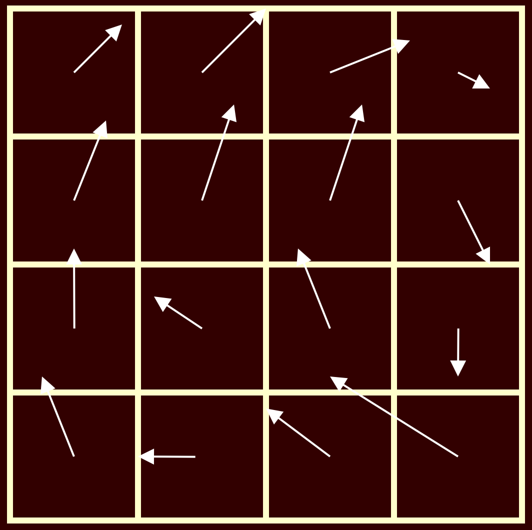
Set interpolated value in new grid

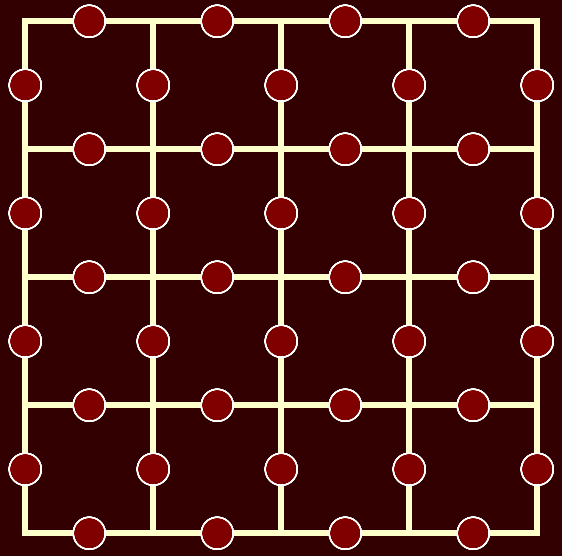
Self-Advection



Repeat for all u-nodes

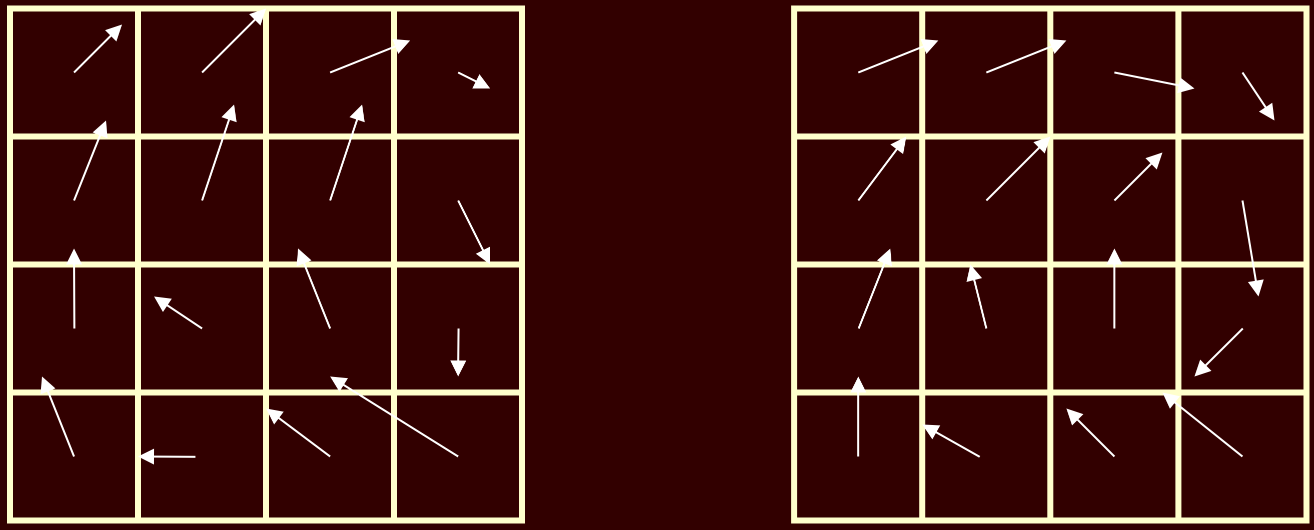
Self-Advection





Similar for v-nodes

Self-Advection



Vmax > Vmax

Advected velocity field

Enforcing Zero Divergence

- Pressure and Velocity fields related
  - Say we have velocity field **w** with non-zero divergence
  - Can decompose into $\mathbf{w} = \mathbf{u} + \nabla p$
    - *Helmholtz-Hodge* Decomposition
    - **u** has zero divergence
  - Define operator P that takes **w** to **u**:

$$\mathbf{u} = P\mathbf{w} = \mathbf{w} - \nabla p$$

  - Apply P to Navier-Stokes Equation:

$$\frac{\partial \mathbf{u}}{\partial t} = P\left(-\left(\mathbf{u} \cdot \nabla\right)\mathbf{u} + v\nabla^2\mathbf{u} + \mathbf{f}\right)$$

    - (Used facts that $P\mathbf{u} = \mathbf{u}$ and $P\nabla p = 0$)

Operator P

- Need to find ∇p
- Implicit definition:

$$\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{u} + \nabla \cdot \nabla p$$
$$\nabla \cdot \mathbf{w} = \nabla^2 p$$

- Poisson equation for scalar field p
  - Neumann boundary condition $\frac{\partial p}{\partial n} = 0$

- Sparse linear system when discretized

Adding Viscosity – Diffusion

- Standard diffusion equation

$$\frac{\partial \mathbf{w}_2}{\partial t} = \nu \nabla^2 \mathbf{w}_2$$

- Use implicit method:

$$\mathbf{w}_3 - \Delta t \frac{\partial \mathbf{w}_3}{\partial t} = \mathbf{w}_2$$

$$\left(\mathbf{I} - \nu \Delta t \nabla^2\right) \mathbf{w}_3 = \mathbf{w}_2$$

- Sparse linear system

Step 4 - Projection

- Enforces mass-conservation condition $\nabla \cdot \mathbf{u} = 0$

- Poisson Problem:

$$\nabla^2 q = \nabla \cdot \mathbf{w}_3 \qquad \mathbf{w}_4 = \mathbf{w}_3 - \nabla q$$

- Discretize q using central differences
  - Sparse linear system
  - Maybe banded diagonal…

- Relaxation methods too inaccurate
  - Method of characteristics more precise for divergence-free field

Solving the System

- Need to calculate: $\frac{\partial \mathbf{u}}{\partial t} = P\left(-\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} + v\nabla^2\mathbf{u} + \mathbf{f}\right)$

- Start with initial state $\mathbf{w}_0 = \mathbf{u}(\mathbf{x}, t)$

- Calculate new velocity fields

$$\mathbf{w}_0 \overbrace{\longrightarrow}^{\text{add force}} \mathbf{w}_1 \overbrace{\longrightarrow}^{\text{advect}} \mathbf{w}_2 \overbrace{\longrightarrow}^{\text{diffuse}} \mathbf{w}_3 \overbrace{\longrightarrow}^{\text{project}} \mathbf{w}_4$$

- New state: $\mathbf{u}(\mathbf{x}, t + \Delta t) = \mathbf{w}_4$

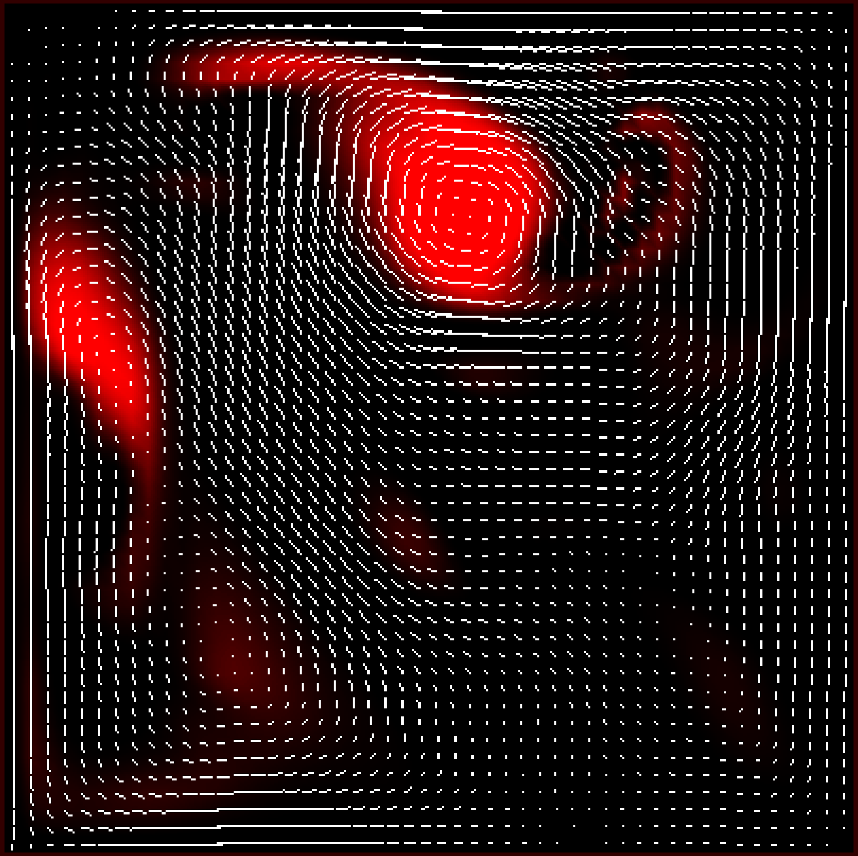
Vorticity Confinement

Basic idea:

Add energy lost as an external force.
Avoid very quick dissipation.

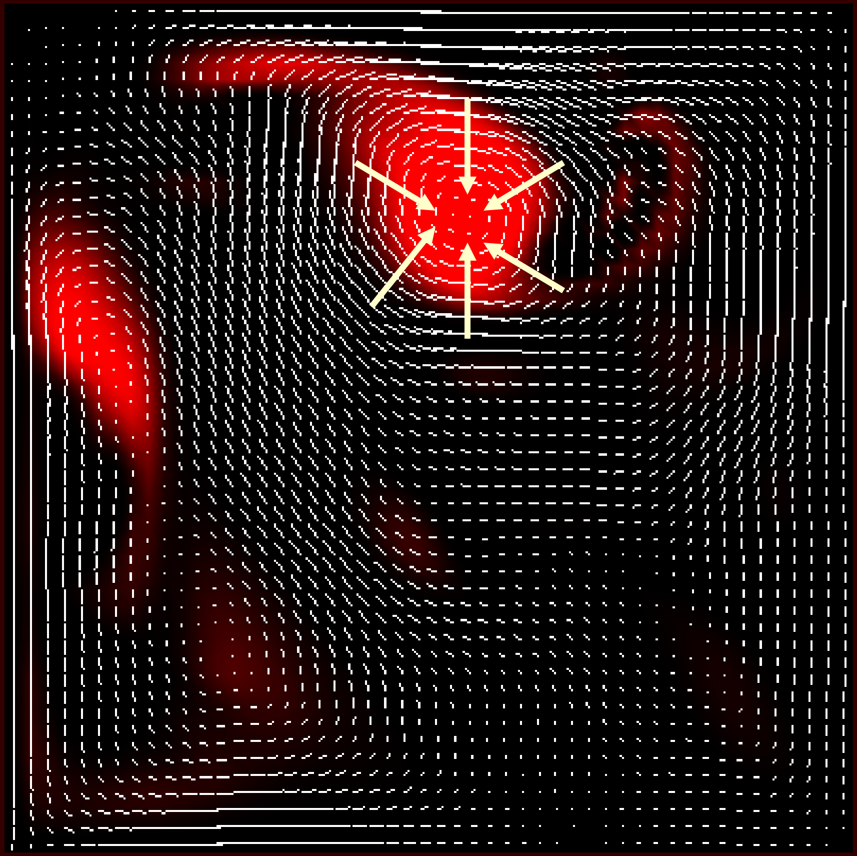
"Vorticity Confinement" force preserves swirling nature of fluids.

Vorticity Confinement



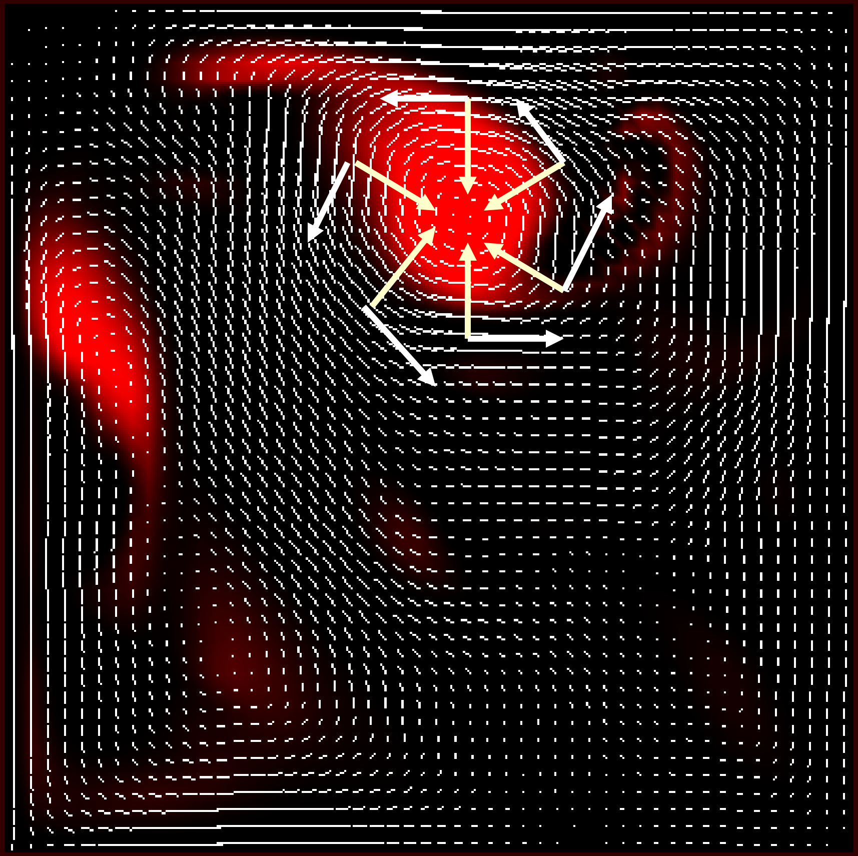
$$\omega = \nabla \times \mathbf{u}$$

Vorticity Confinement



$$\mathbf{N} = \frac{\eta}{|\eta|} \quad \eta = \nabla|\omega|$$

Vorticity Confinement



$$\mathbf{f} = \epsilon \, h \, (\mathbf{N} \times \omega)$$

Videos