Stacking Dominoes with Imperfect Positioning
16-741 Project Final Report

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Abstract

A problem of stacking dominoes on top of each other with only one piece at the lowest level so as to achieve maximum overhang is a well-known mathematical puzzle [1]. Several solutions exist. The most popular solution achieves overhang of H_n for n dominoes, where H_n is n-th harmonic number [1, 2]. There also other solutions that achieve greater overhang for the same number of dominoes [3]. However, the problem of assembling these structures is usually not considered. This paper concentrates on assembling the structure from bottom up with an imperfect manipulator and placing one domino piece at a time. Under reasonble assumptions it shows that the harmonic series solution is the only one that can tolerate manipulation imperfections and investigates the problem experimentally.

1 Problem formulation

The problem can be formulated as follows in plain English [2]: *"When we stack dominoes with only one touching the table, what is the greatest horizontal distance we can cover keeping in balance?"*. More formally, we have the following conditions:

1. There are n identical domino pieces each of length 2. Each piece has mass uniformly distributed along the length.
2. The dominoes must be stacked flat on top of each other. Also it is not allowed to rotate them in the horizontal plane (otherwise we could make the corner of a domino extend further from the base). The latter condition makes the problem "1.5-dimensional" (1 horizontal dimension with values in $\mathbb{R}$ and 1 vertical dimension with values in $\mathbb{N}$). It also makes the results independent of height and width of the dominoes.
3. The problem is to find values $\{a_1, m_1\}, \ldots, \{a_n, m_n\}$, where $a_k \in \mathbb{R}$ is a horizontal displacement of k-th piece (see Fig. 1 for the coordinate system) and $m_k \in \mathbb{N}$ is its level in the stack - pieces on level m are higher than those on level $m-1$. We need to maximize

$$\max_{k=1\ldots n} a_k$$

subject to these constraints: the resulting structure is in balance given gravitational forces and only m_1 has value of 1.

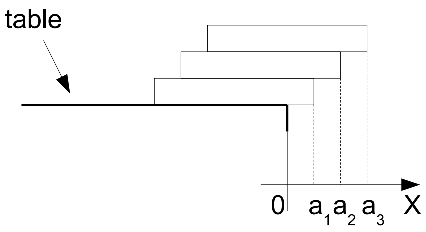


Figure 1: The problem formulation. Maximum overhang achieved by this placement is a_3.

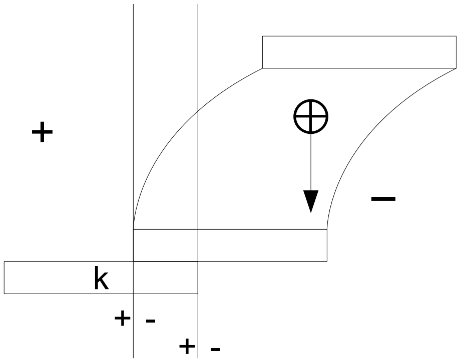
A surprising result is that given sufficiently large number of dominoes, arbitrary large overhangs can be achieved.

2 Stability criterion

If each domino in the structure rests on only one other domino, there is a simple recursive criterion to determine whether the structure is in balance: for each piece k the following should hold:

$$a_{cm}(k) \in [a_k - 1, a_k + 1], \tag{1}$$

where $a_{cm}(k)$ is the X coordinate of the center of mass of the part of the structure that rests on domino k. There is a moment-labeling argument for this criterion - assuming that the part of the structure that rests on k is balanced, it acts like a rigid body and we can apply moment labeling to the combination of that part and domino k:



In order for the reaction force from k to be able to compensate the resultant gravitational force acting on the part above k, the line of action of gravitational force must be in the unlabeled part of the space, which exactly corresponds to Equation 1. In order for the whole structure to be balanced, the criterion must hold for all of its parts.

3 Known solutions

3.1 Harmonic series

This is the most well-known solution [1, 2, 3]. If we restrict the structures to only have 1 domino piece per horizontal layer (set $m_k = k$), then the following condition ensures the optimality: the center of mass of a block consisting of pieces $k, \ldots, n$ must lie exactly above the right edge of piece $k-1$. Formally we have

$$\frac{1}{n-k+1}\sum_{i=k}^{n}(a_i-1) = a_{k-1} \tag{2}$$

Theorem 1 *The structure with maximum overhang consisting of n dominoes with 1 domino per horizontal levels has horizontal domino coordinates obeying Equation 2.*

Proof: Suppose $\exists k$:

$$\frac{1}{n-k+1}\sum_{i=k}^{n}(a_i-1) \neq a_{k-1}.$$

If

$$\frac{1}{n-k+1}\sum_{i=k}^{n}(a_i-1) > a_{k-1},$$

then the part consisting of pieces $k, \ldots, n$ cannot be in balance on piece $k-1$ and will fall. Hence we can only have

$$\frac{1}{n-k+1}\sum_{i=k}^{n}(a_i-1) < a_{k-1}$$

Let us assume that the whole structure is balanced. Then we can achieve a larger overhang by moving the part consisting of pieces $k, \ldots, n$ to the right by $\Delta_1 = \frac{l}{n-k+2}$ and piece $k-1$ by $\Delta_2 \frac{(n-k+1)l}{n-k+2}$ to the left, where

$$l = a_{k-1} - \frac{1}{n-k+1}\sum_{i=k}^{n}(a_i-1)$$

is the distance along X axis between the center of gravity of $k, \ldots, n$ component and the edge of $(k-1)$-th piece. Because

$$(m-k+1)\Delta_1 = \Delta_2,$$

the center of gravity of any component $k-p, \ldots, n$ $\forall p$ does not move. The $k, \ldots, n$ component is balanced on the edge of $(k-1)$-th piece. Therefore the new structure is balanced and the total overhang has been increased by Δ_1. $\Box$

From Equation 2 we can deduce the following recursive relation between a_k and a_{k+1}:

$$\begin{aligned}
a_k &= \frac{1}{n-k}\sum_{i=k+1}^{n}(a_i-1) \\
&= \frac{a_{k+1}-1}{n-k} + \frac{1}{n-k}\sum_{i=k+2}^{n}(a_i-1) \\
&= \frac{a_{k+1}-1}{n-k} + \frac{n-k-1}{n-k}a_{k+1} \\
&= a_{k+1} - \frac{1}{n-k}
\end{aligned}$$

The total overhang achieved by this structure is thus equal to

$$\sum_{k=0}^{n-1}(a_{k+1}-a_k) = \sum_{k=0}^{n-1}\frac{1}{n-k} = \sum_{k=1}^{n}\frac{1}{k} = H_n$$

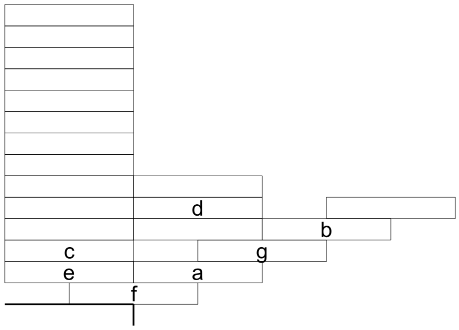


Figure 2: Structure with counterweight

where H_n is the n-th harmonic number. $a_0 = 0$ is the position of the table.

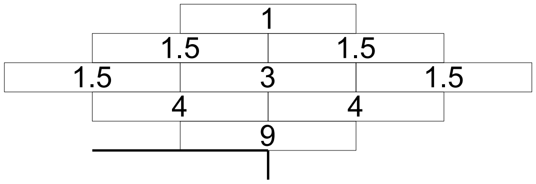
The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is diverging, so given a large enough number of dominoes and perfect manipulator we could assemble a structure with arbitrary large overhang (however, we would need to choose the overhang before starting the construction - it is not possible to go on building it indefinitely long with larger and larger overhang). The growth rate of H_n is slow, because [1]

$$\ln n < H_n < 1 + \ln n$$

so we need an exponential number of dominoes to create an overhang of given length.

3.2 Cantilever structure

If we do not restrict the structures to have exactly one domino per horizontal level, we can achieve larger overhangs for the same values of n. One example is a cantilever structure [2]:



The structure of height $2n - 1$ achieves an overhang of k and requires

$$n + 2\sum_{k=1}^{n-1} k = n^2$$

dominoes. Therefore the overhang for n dominoes is approximately $\sqrt{n}$, which grows much faster than H_n.

The analysis of whether the structure is balanced is more involved if there are dominoes that rest on two other dominoes as in the case with cantilever structure. We need to consider the distribution of forces along the contact surface. I will not list the complete proof, but for the pictured example the proof is simple enough - on the picture each piece is marked with a number. This number denotes the magnitude of force with which that piece acts on the pieces below it. One can deduce from the magnitudes the distribution of forces along the contact surfaces and convince oneself that the system is indeed in balance.

3.3 Structure with counterweight

Another variation is the structure with counterweight in Fig. 2 (taken from [3]). The reader will argue that the structure is obviously not in balance - pieces a and b are going to fall. However, this problem can be remedied by moving c and d by infinitisemal distances to the right so that they hold a and b in place.

Unlike the cantilever structure, this one needs an exponential number of dominoes to achieve an overhang of given length. To see this, first observe that 1 domino trivially achieves an overhang of 1. Let us suppose we have a structure with overhang $2k+1$, for example the part of the structure consisting of piece g and everything resting on it from Fig. 2. To increase the overhang by 2, place it on top of 3-piece structure (aef in the figure), shift right appropriately and add enough pieces on e for counterweight. If we had N_{2k+1} pieces in the original structure, we will need $2N_{2k+1}$ for counterweight, so

$$N_{2k+3} = 2N_{2k+1} + 3 + N_{2k+1} = 3(N_{2k+1} + 1)$$

It is easy to see that

$$N_{2k+1} = \frac{5}{2}3^k - \frac{3}{2}$$

satisfies this recurrence and the initial condition $N_1 = 1$, so the number of dominoes that we need indeed grows exponentially with the overhang. Therefore it does not have an advantage over the harmonic series solution in this respect.

4 Imperfect manipulator model

The previous section listed several solutions, but did not show a way to assemble any of those structures. The possibility of assembling them depends on our assumptions about the manipulator to be used for assembly. This paper uses the following assumptions:

1. The manipulator operates one domino piece at a time.
2. It is only able to place the piece on top of the table or other pieces, i.e. it cannot "insert" it into a slot formed by other pieces.
3. The vertical motion of the manipulator and placement of the piece is precise and quasistatic.
4. The horizontal error of the placement has Gaussian distribution with mean 0 and variance σ^2.
5. Once a domino is released by the manipulator, it cannot be moved, i.e. the manipulator gets to move each piece exactly once and cannot correct errors in placement.

Note that these assumptions are quite restrictive from the point of view of what we can achieve with just our two hands. For example, the structure in Fig. 3 cannot be assembled with such a manipulator, although there is an easy way to assemble it in principle: make a vertical stack of pieces $1, \ldots, 8$, then holding 1 and 3 in place pull 2 a little to the left, insert 9 in the resulting slot, and finally pull 2 to the left for the remaining distance. Our manipulator would need to place 2 and 9 right after 1, and without the counterbalance provided by $3, \ldots, 8$ they will fall.

5 Uniqueness of the harmonic series solution

The following simple result states that any structure having more than 1 domino on some level, i.e. having $m_{k_1} = m_{k_2}, k_1 \neq k_2$ will fall apart almost surely as we try to assemble it with an imperfect manipulator from the previous section.

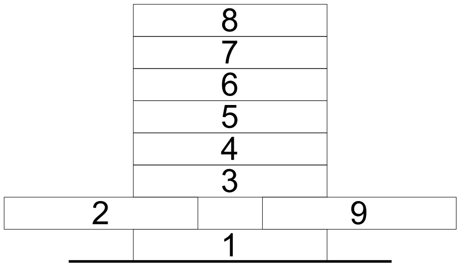


Figure 3: This stack is impossible to put together with a manipulator considered in the paper.

Theorem 2 *Let the balanced structure* $\{a_1, m_1\}, \ldots, \{a_n, m_n\}$ *have* $m_{k_1} = m_{k_2} > 1$ *for some* $k_1 \neq k_2$. *Under the assumptions of the previous section the manipulator will fail to assemble this structure with probability 1. (Actually, we can relax the assumption of Gaussian horizontal noise - the result will hold for noise having any PDF with finite values).*

Proof: Consider the smallest m such that $m_{k_1} = m_{k_2} = m$ for some $k_1 \neq k_2$. The pieces k_1 and k_2 must be on the same level and level $m - 1$ has only one domino. Denote the index of that domino k_3. Then both k_1 and k_2 must be placed on top of k_3 and must be in balance right after placement. The only way (up to swapping labels a_{k_1} and a_{k_2}) to achieve that is to have

$$a_{k_1} = a_{k_3} + 1, \;\; a_{k_2} = a_{k_3} - 1,$$

that is the centers of gravity of two pieces to be exactly above the edges of the underlying piece. In order for them to be in balance, they need to be placed without any error along X axis - otherwise at least one domino will extend more than 1 unit beyond the corresponding edge of k_3 and fall. But

$$P(a'_{k_1} = a_{k_1}, a'_{k_2} = a_{k_1}) = 0$$

because the PDF of manipulator error is finite. (a' is the location of a domino resulting from manipulation). Thus with probability one the assembly process will fail. $\Box$

The theorem means that although there are structures that achive a much larger overhang given the same number of dominoes than the harmonic series solution, none of them can withstand even slight imperfections in manupulation[1]. For the harmonic series solution, however, there is a nonzero probablity that all n pieces placed by the manipulator will be balanced, although with probability 1 the total overhang will not reach the theoretical maximum of H_n.

6 Experiments

This section presents the results of a simulation of an imperfect manipulator assembling the harmonic series structure. The number of dominoes ws fixed to be 100. The maximum overhang achieveable with perfect manipulation is thus

$$H_{100} \approx 5.187$$

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[1] We can also try and modify the horizontal positions of the items somewhat to try and improve the probability of successfully completing the structure or the expected overhang achieved before the failure - see the experiments section for this. However the uniqueness of the solution is in that we always need to have only 1 piece per horizontal layer.

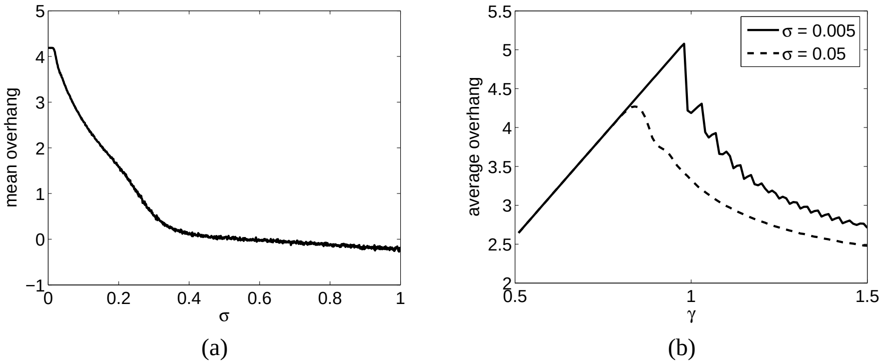


Figure 4: Results for manipulation without information about the location of already placed pieces.

For each simulated assembly process I measured the overhang achieved just before the structure crashed because of lack of balance, i.e. if it crashed when the manipulator was trying to put 98-th domino down, the overhang achieved by the first 97 was measured. The mean overhang is the average over multiple randomized trials using identical model parameters. The individual experiments are described in more detail in their respective subsections.

6.1 Basic case - sensorless manipulation

In this experiment I assumed that the manipulator does not know where it had put the pieces that are already in place (i.e. what are their displacements due to manipulation error) and tries to put the next piece in theoretically optimal for perfect manipulation position, i.e. it tries to achieve

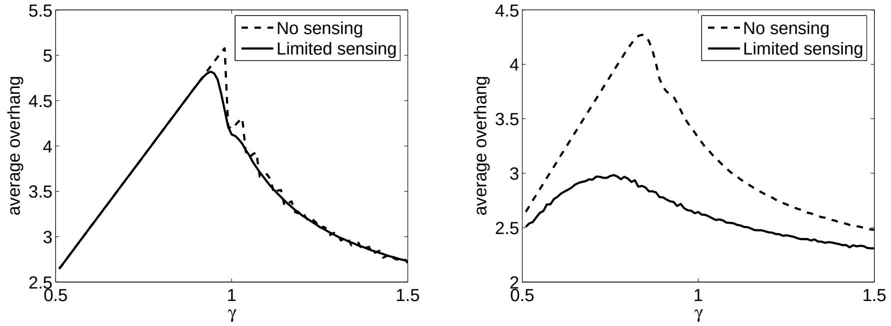
$$a_k = \sum_{i=1}^{k} \frac{1}{n-i+1}$$

The noise σ of the manipulator was varied from 0.001 to 1 length unit (the length of a domino is 2 units). For each noise level 1000 random iterations were made. The averaged results are presented in Fig. 4(a). One can see that the overhang decreases as noise increases. It is exactly what one would expect - the worse is the manipulator precision, the worse is its performance in building a delicate balancing structure.

One thing to notice about this experiment is that even for the smallest noise value the average overhang is approximately 1 unit shorter than the theoretical maximum. The reason for that is that the last piece almost always creates imbalance in some part of the structure and the first 99 pieces are almost always balanced, because there is so much “reserve” in the perfect structure to accomodate the upper piece. Even if a small amount of this “reserve” is taken by errors in manipulation, usually it is enough to still have a balanced structure until the attempt to put the 100th piece in its place.

6.2 Taking noise into account - sensorless case

One way to take imperfections of the manipulator into account is to not try to aim the dominoes for the positions that are optimal in the precise manipulator case. In this experiment the manipulator still did not know the offsets of the pieces that it has already placed. The



(a) (b)

Figure 5: Limited sensing case results for $\sigma = 0.005$ (left) and $\sigma = 0.05$ (right)

target position of k-th piece was

$$a_k(\gamma) = \gamma \sum_{i=1}^{k} \frac{1}{n-i+1}$$

where factor γ was varied from 0.5 to 1.5. For each value of γ 5000 random iterations were made. The results for $\sigma = 0.005$ and $\sigma = 0.05$ are in Fig. 4(b).

One can see that the optimal γ is less than 1 (i.e. with imperfect manipulator we should not try to place pieces as aggresively to the right as with a perfect one), and also that it depends on manipulator noise value. Using optimal γ for $\sigma = 0.005$ we are able to achive average overhangs close the theoretical for maximum for precise manipulation. Also by the locations of the peaks in the plot corresponding to $\sigma = 0.005$ one can see the values of γ for which a transition is made from structures with k pieces being usually stable to being usually unstable - each peak denotes a decrement of k by 1.

6.3 Taking noise into account - limited sensing case

“Limited” in the section title refers not to sensing, but rather to the way we deal with the sensing results. Assume that the manipulator now knows about the real positions of the pieces that it has already place. Then one approach would be to aim for the following location of the next piece:

$$a_k(\gamma, a_{k-1}) = a_{k-1} + \gamma \frac{1}{n-k+1}$$

Again, 5000 trials were done for each combination of parameter values. The results are in Fig. 5(a) ($\sigma = 0.005$) and Fig. 5(b) ($\sigma = 0.05$). One can see that the idea is not a good one - the maximal overhang is smaller (especially for $\sigma = 0.05$) than one without taking the real positions of the pieces into account. Of course it does not suggest that the information about the real positions is useless for this problem, only that this particular way of using it is the wrong one.

7 Conclusion

This paper considered a well-known puzzle of stacking dominoes on top of each other so as to achieve a maximal overhang. It emphasized the problem of assembling the solution to the problem with an imperfect manipulator. It has shown that for a certain class of manipulator only the solutions with 1 domino per horizontal level are feasible. Also it has demonstrated that close to theoretically optimal overhangs can be achieved if the precision of the manipulator is good enough and the fact that the manipulator is not perfect is taken into account. The problem of finding the optimal way to take the positions of already placed dominoes so as to maximize overhang remains to be solved.

References

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[2] All a matter of balance - or a problem with dominoes. C. Sangwin. 2003

[3] A domino overhang. `http://www.ken.duisenberg.com/potw/archive/arch03/030728sol.html`