# Lecture 8
# Quaternions

Matthew T. Mason

Mechanics of Manipulation

# Today's outline

Overview, motivation

Background

Definition and properties

Rotation using unit quaternions

Intuition

Using quaternions to represent rotations

Why we love quaternions.

# Motivation

## Motivation

- Quaternions have nice geometrical interpretation.
- Quaternions have advantages in representing rotation.
- Quaternions are cool. Even if you don't want to use them, you might need to defend yourself from quaternion fanatics.

# Why can't we invert vectors in $\mathbf{R}^3$?

- We can invert reals. $x \times \frac{1}{x} = 1$.
- We can invert elements of $\mathbf{R}^2$ using complex numbers. $z \times z^*/|z|^2 = 1$, where $^*$ is complex conjugate.
- Can we invert $\mathbf{v} \in \mathbf{R}^3$?
  - No.
- How about $\mathbf{v} \in \mathbf{R}^4$?
  - Yes! Hamilton's quaternions are to $\mathbf{R}^4$ what complex numbers are to $\mathbf{R}^2$.

# Complex numbers versus quaternions





## Definition (Complex numbers)

- Define basis elements 1 and $i$;
- Define complex numbers as a vector space over reals: elements have the form $x + iy$;
- One more axiom required: $i^2 = -1$.

## Definition (Quaternions)

- Define basis elements 1, $i$, $j$, $k$;
- Define quaternions as a vector space over reals: elements have the form $q_0 + q_1 i + q_2 j + q_3 k$;
- One more axiom:

$$i^2 = j^2 = k^2 = ijk = -1$$

# Basis element multiplication





From that one axiom, we can derive other products:

$$
\begin{aligned}
ijk &= -1 \\
i(ijk) &= i(-1) \\
-jk &= -i \\
jk &= i
\end{aligned}
$$

Writing them all down:

$$
\begin{aligned}
ij &= k, ji = -k \\
jk &= i, kj = -i \\
ki &= j, ik = -j
\end{aligned}
$$

Quaternion products of $i, j, k$ behave like cross product.

# Quaternion notation





We can write a quaternion several ways:

$$q = q_0 + q_1 i + q_2 j + q_3 k$$
$$q = (q_0, q_1, q_2, q_3)$$
$$q = q_0 + \mathbf{q}$$

## Definition (Scalar part; vector part)

For quaternion $q_0 + \mathbf{q}$, $q_0$ is the scalar part and $\mathbf{q}$ is the vector part

# Quaternion product





We can write a quaternion product several ways:

$$
\begin{aligned}
pq &= (p_0 + p_1 i + p_2 j + p_3 k)(q_0 + q_1 i + q_2 j + q_3 k) \\
&= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) + \ldots i + \ldots j + \ldots k \\
pq &= (p_0 + \mathbf{p})(q_0 + \mathbf{q}) \\
&= (p_0 q_0 + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{pq})
\end{aligned}
$$

The last product includes many different kinds of product: product of two reals, scalar product of vectors. But what is **pq**? Cross product? Dot product? Both! Cross product minus dot product!

$$pq = (p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})$$

# Conjugate, length

## Definition (Conjugate)

$$q^* = q_0 - q_1 i - q_2 j - q_3 k$$

Note that

$$\begin{aligned} qq^* &= (q_0 + \mathbf{q})(q_0 - \mathbf{q}) \\ &= q_0^2 + q_0\mathbf{q} - q_0\mathbf{q} - \mathbf{q}\mathbf{q} \\ &= q_0^2 + \mathbf{q} \cdot \mathbf{q} - \mathbf{q} \times \mathbf{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \end{aligned}$$

## Definition (Length)

$$|q| = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

# Quaternion inverse

Every quaternion except 0 has an inverse:

$$q^{-1} = \frac{q^*}{|q|^2}$$

Without commutativity, quaternions are a *division ring*, or a *non-commutative field*, or a *skew field*.
Just as complex numbers are an extension of the reals, quaternions are an extension of the complex numbers (and of the reals).
If 1D numbers are the reals, and 2D numbers are the complex numbers, then 4D numbers are quaternions, and that's all there is. (Frobenius) (Octonions are not associative.)

# Rotation using unit quaternions





- Let $q$ be a unit quaternion, i.e. $|q| = 1$.
  - It can be expressed as

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

- Let $x = 0 + \mathbf{x}$ be a "pure vector".
- Let $x' = qxq^*$.
- Then $x'$ is the pure vector $\mathrm{rot}(\theta, \hat{\mathbf{n}})\mathbf{x}$!!!

# Proof that unit quaternions work

- Expand the product $qxq^*$;
- Apply half angle formulas;
- Simplify;
- Compare with Rodrigues's formula.

Sadly, not all proofs confer insight.

# Why $\theta/2$? Why $qxq^*$ instead of $qx$?





In analogy with complex numbers, why not use

$$p = \cos\theta + \hat{\mathbf{n}}\sin\theta$$
$$\mathbf{x}' = p\mathbf{x}$$

To explore that idea, define a map $L_p(q) = pq$. Note that $L_p(q)$ can be written:

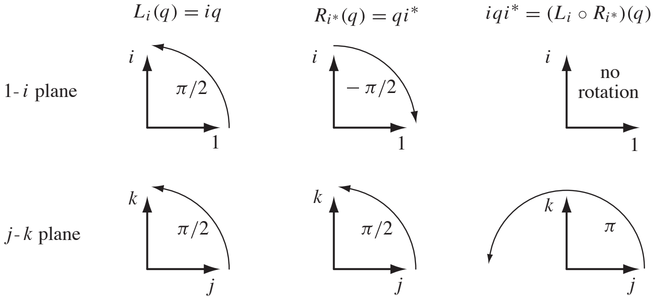
$$L_p(q) = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Note that the matrix above is orthonormal. $L_p$ is a rotation of Euclidean 4 space!

# Geometrical explanation

Although $L_p(q)$ rotates the 4D space of quaternions, it is *not* a rotation of the 3D subspace of pure vectors. Some of the 3D subspace leaks into the fourth dimension.

Consider an example using $p = i$. Is it a rotation about $i$ of $\pi/2$?



# What do we do with a representation?

Rotate a point: $qxq^*$.

Compose two rotations:

$$q(p\mathbf{x}p^*)q^* = (qp)\mathbf{x}(qp)^*$$

Convert to other representations:

- From axis-angle to quaternion:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

- From quaternion to axis-angle:

$$\theta = 2\tan^{-1}(|\mathbf{q}|, q_0)$$
$$\hat{\mathbf{n}} = \mathbf{q}/|\mathbf{q}|$$

assuming $\theta$ is nonzero.

# From quaternion to rotation matrix





Just expand the product

$$qxq^* = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \mathbf{x}$$

# From rotation matrix to quaternion



Given $R = (r_{ij})$, solve expression on previous slide for quaternion elements $q_i$

Linear combinations of diagonal elements seem to solve the problem:

$$
\begin{aligned}
q_0^2 &= \frac{1}{4}(1 + r_{11} + r_{22} + r_{33}) \\
q_1^2 &= \frac{1}{4}(1 + r_{11} - r_{22} - r_{33}) \\
q_2^2 &= \frac{1}{4}(1 - r_{11} + r_{22} - r_{33}) \\
q_3^2 &= \frac{1}{4}(1 - r_{11} - r_{22} + r_{33})
\end{aligned}
$$

so take four square roots and you're done? You have to figure the signs out. There is a better way . . .

# Look at the off-diagonal elements

- $$\begin{aligned} q_0 q_1 &= \frac{1}{4}(r_{32} - r_{23}) \\ q_0 q_2 &= \frac{1}{4}(r_{13} - r_{31}) \\ q_0 q_3 &= \frac{1}{4}(r_{21} - r_{12}) \\ q_1 q_2 &= \frac{1}{4}(r_{12} + r_{21}) \\ q_1 q_3 &= \frac{1}{4}(r_{13} + r_{31}) \\ q_2 q_3 &= \frac{1}{4}(r_{23} + r_{32}) \end{aligned}$$
- Given any one $q_i$, could solve the above for the other three.

# The procedure

1. Use first four equations to find the largest $q_i^2$. Take its square root, with either sign.
2. Use the last six equations (well, three of them anyway) to solve for the other $q_i$.

- That way, only have to worry about getting one sign right.
- Actually $q$ and $-q$ represent the same rotation, so no worries about signs.
- Taking the largest square root avoids division by small numbers.

# Properties of unit quaternions





- Unit quaternions live on the unit sphere in $\mathbf{R}^4$.
- Quaternions $q$ and $-q$ represent the same rotation.
- Inverse of rotation $q$ is the conjugate $q^*$.
- Null rotation, the identity, is the quaternion 1.

# Metric on SO(3)

What is the right metric between two spatial rotations? I.e. between two points in $SO(3)$?

I provided an answer in Lecture 2! $SO(3)$ is a configuration space. Let $R_1$ and $R_2$ be two displacements of the sphere $S^2$. Let $x$ by a point on the sphere. Let $\alpha$ be the usual metric on the sphere—the length of the shortest great circle arc joining two points. Define $d(R_1, R_2) = \max_{x \in S^2} \alpha(R_1(x), R_2(x))$. That is equivalent to taking the rotation angle of $R_1^{-1} R_2$. In brief, the angle of the rotation that takes you from $R_1$ to $R_2$.

# Metric on sphere of unit quaternions

- Consider unit quaternion

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

  Shortest path on the unit sphere joining $\pm q$ with 1 has length $\theta/2$.
- What is the shortest distance on the unit quaternion sphere $S^3$ from $\pm p$ to $\pm q$?
- Multiplication by a unit quaternion is a rotation of the unit quaternion sphere. I.e. distance-preserving. So the distance from $\pm p$ to $\pm q$ is the same as the distance from $\pm pq^*$ to 1. I.e. $\alpha/2$, where $\alpha$ is the rotation angle required from $p$ to $q$.
- So usual metric on the sphere, applied to unit quaternions, is the right metric for $SO(3)$. (Factor of two is irrelevant.)

# Topology of $SO(3)$





The right metric matters. Our representation is smooth and one-to-one, apart from the fact that antipodes represent the same rotation. That means the unit quaternions, with antipodes identified, have the same topology as $SO(3)$. That's how we know the topology of $SO(3)$ is Projective 3-space, $\mathbb{P}^3$.

# Measure, probabilities, sampling

- Likewise we can turn our attention to *measure*.
- Easiest way is to think in terms of probabilities. Suppose you are given some probability density function on $SO(3)$. How can you decide whether it is unbiased?
- Pick some subset of $SO(3)$, integrate the probability of that subset. Now rigidly rotate that subset. If the probability is invariant with respect to rotation, the PDF is unbiased. A measure with this property is the Haar measure, and it is uniquely determined up to some scale factor.
- Look at a uniform density function on the unit quaternion sphere $S^3$. It has the right property — it is invariant with respect to rotations. The quaternion representation gives you the right measure, and is an easy way to correctly sample $SO(3)$.