Lecture 9.
Representing constraint

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Mechanics of Manipulation
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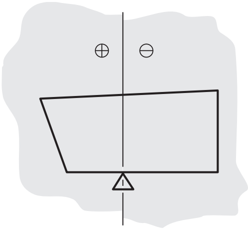
Today's outline

Analyzing unilateral constraint.

- Graphical technique: (Reauleaux)
  - Construct contact normal;
  - Label left side "$+$";
  - Label right side "$-$";
  - Label normal "$\pm$";
- Great for humans.
- Bad for computers.
- Doesn't extend to higher dimensions.
- *We need an analytical technique!*



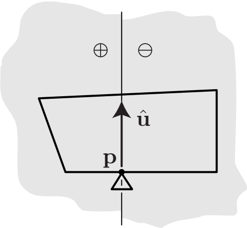
First order model of constraint

- Let $\hat{\mathbf{u}}$ be contact normal, inward pointing
- Let p be contact point in the constrained body
- Let $\mathbf{v}_p$ be the velocity of the point p.
- Then we write the bilateral velocity constraint as

$$\hat{\mathbf{u}} \cdot \mathbf{v}_p = 0$$

and unilateral velocity constraint as

$$\hat{\mathbf{u}} \cdot \mathbf{v}_p \geq 0$$



Express as reciprocal product

- Let $(\omega, \mathbf{v}_0)$ be screw coordinates of body velocity
- Then velocity of $\mathbf{p}$ is

$$\mathbf{v}_p = \mathbf{v}_0 + \omega \times \mathbf{p}$$

- We write the kinematic constraint

$$\hat{\mathbf{u}} \cdot (\mathbf{v}_0 + \omega \times \mathbf{p}) \geq 0$$

- Distribute dot product, play with triple product ...

$$\hat{\mathbf{u}} \cdot \mathbf{v}_0 + (\mathbf{p} \times \hat{\mathbf{u}}) \cdot \omega \geq 0$$

- This looks like a reciprocal product. We just need to define a new screw.

Contact screw

Definition (Contact screw)

Define the contact screw to be the contact normal, with pitch zero.

- Then the screw coordinates of the contact screw are just the Plücker coordinates of the contact normal :

$$(\mathbf{c}, \mathbf{c}_0) = (\mathbf{u}, \mathbf{p} \times \hat{\mathbf{u}})$$

- and we can write the unilateral constraint as

$$(\mathbf{c}, \mathbf{c}_0) * (\omega, \mathbf{v}_0) \geq 0$$

Reciprocal, contrary, repelling

Definition (Reciprocal, contrary, repelling)

A pair of screws is reciprocal, contrary, or repelling, if their reciprocal product is zero, negative, or positive, respectively.

- The bilateral constraint is satisfied if the velocity screw $(\omega, \mathbf{v}_0)$ and the contact screw $(\mathbf{c}, \mathbf{c}_0)$ are *reciprocal*:

$$(\mathbf{c}, \mathbf{c}_0) * (\omega, \mathbf{v}_0) = 0$$

- The unilateral constraint is satisfied if the velocity screw $(\omega, \mathbf{v}_0)$ and the contact screw $(\mathbf{c}, \mathbf{c}_0)$ are *reciprocal or repelling*:

$$(\mathbf{c}, \mathbf{c}_0) * (\omega, \mathbf{v}_0) \geq 0$$

Connection to Reuleaux's method

- Contact screw:
  - Analytically: $(\mathbf{c}, \mathbf{c}_0)$.
  - Graphically: the contact normal.

Copy the figure from the board

- Velocity twist:
  - Analytically: $(\omega, \mathbf{v}_0)$.
  - Graphically: labelled IC. "$+$" means ω is parallel to $+z$. $-$ means ω is parallel to $-z$.
- Reciprocal or repelling:
  - Analytically: non-negative virtual product.
  - Graphically: "$+$" IC on the left, "$-$" on the right, or either label if coincident with the contact normal.

Example 1

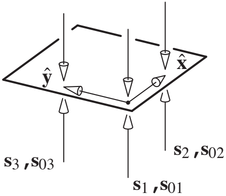
Problem

How are *planar* motions represented using screw coordinates?

- We developed screw coordinates, in part, so that we could deal with three dimensional motions.
- Nonetheless, planar motions are an important subgroup of spatial motions.
- Obviously we should need only three parameters. Does this map nicely onto screw coordinates? If so, which three parameters is it?
- How can we address the problem

Ex 1: screw coordinates of planar motions

- Solution method: use the machinery we just developed for analyzing constraints.
- Choose three bilateral constraints to leave only planar motions.
- Write the screw coordinates for the constraints. Use cross products, or just write them down by inspection.



$$(\mathbf{s}_1, \mathbf{s}_{01}) = (0, 0, 1, 0, 0, 0)$$
$$(\mathbf{s}_2, \mathbf{s}_{02}) = (0, 0, 1, 0, -1, 0)$$
$$(\mathbf{s}_3, \mathbf{s}_{03}) = (0, 0, 1, 1, 0, 0)$$

- Let the twist be given by

$$(\mathbf{t}, \mathbf{t}_0) = (t_1, t_2, t_3, t_4, t_5, t_6)$$

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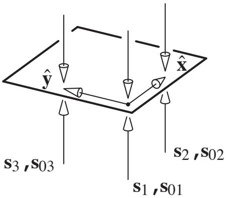

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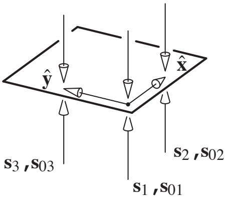

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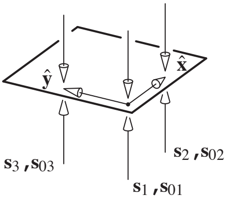

$$
\begin{aligned}
(\mathbf{s}_1, \mathbf{s}_{01}) &= (0, 0, 1, 0, 0, 0) \\
(\mathbf{s}_2, \mathbf{s}_{02}) &= (0, 0, 1, 0, -1, 0) \\
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\end{aligned}
$$

- Let the twist be given by

$$(\mathbf{t}, \mathbf{t}_0) = (t_1, t_2, t_3, t_4, t_5, t_6)$$

Ex 1: Form reciprocal products.

- The twist must be reciprocal to $(\mathbf{s}_1, \mathbf{s}_{01})$:

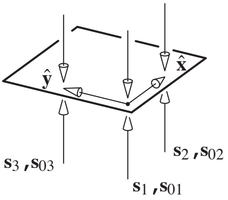
$$t_6 = 0$$

- ... to $(\mathbf{s}_2, \mathbf{s}_{02})$:

$$t_6 - t_2 = 0$$

- ... and to $(\mathbf{s}_3, \mathbf{s}_{03})$:

$$t_6 + t_1 = 0$$



- Thus the twist must be of the form

$$(\mathbf{t}, \mathbf{t}_0) = (0, 0, t_3, t_4, t_5, 0)$$

Ex 1: Interpreting the answer analytically

- The twist must be of the form

$$(\mathbf{t}, \mathbf{t}_0) = (0, 0, t_3, t_4, t_5, 0)$$

- By comparing with the usual screw coordinates of a twist $(\omega, \mathbf{v}_0)$,

$$\omega = (0, 0, t_3)$$
$$\mathbf{v}_0 = (t_4, t_v, 0)$$

so the planar motion is represented by its angular velocity and the velocity at the origin—three parameters as expected.

Ex 1: Interpreting the answer geometrically

- The screw coordinates are:

$$(\mathbf{t}, \mathbf{t}_0) = (0, 0, t_3, t_4, t_5, 0)$$

- To get the pitch:

$$p = \frac{\mathbf{t} \cdot \mathbf{t}_0}{\mathbf{t} \cdot \mathbf{t}} = 0$$

- Zero-pitch! The screw coordinates are the Plücker coordinates of a line parallel to z.
- Where does this line pierce the x-y plane? At the point closest to the origin:

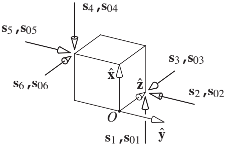
$$\frac{\mathbf{t} \times \mathbf{t}_0}{\mathbf{t} \cdot \mathbf{t}} = (-t_5 t_3, t_4 t_3, 0)/t_3^2$$
$$= (-t_5/t_3, t_4/t_3)$$

  which happens to be the coordinates of the IC.
- So the screw coordinates are the Plücker coordinates of a rotation axis, parallel to the $\hat{\mathbf{z}}$ axis, through the IC. As expected.

Ex 2: Squeezing the corners of a cube

- Suppose six rigid contacts are applied as shown in the figure. Can the cube move?
- We will consider the simpler bilateral problem

$$(\mathbf{s}_1, \mathbf{s}_{01}) = (1, 0, 0, 0, 1, 0)$$
$$(\mathbf{s}_2, \mathbf{s}_{02}) = (0, -1, 0, 1, 0, 0)$$
$$(\mathbf{s}_3, \mathbf{s}_{03}) = (0, 0, -1, 0, 0, 0)$$
$$(\mathbf{s}_4, \mathbf{s}_{04}) = (-1, 0, 0, 0, 0, -1)$$
$$(\mathbf{s}_5, \mathbf{s}_{05}) = (0, 1, 0, 0, 0, 1)$$
$$(\mathbf{s}_6, \mathbf{s}_{06}) = (0, 0, 1, -1, -1, 0)$$



- Let $(\mathbf{t}, \mathbf{t}_0)$ be a differential twist. Reciprocal with respect to $(\mathbf{s}_1, \mathbf{s}_{01})$:

$$t_4 + t_2 = 0$$

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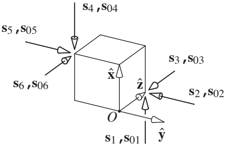

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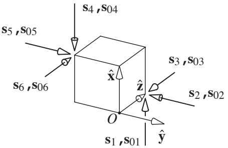


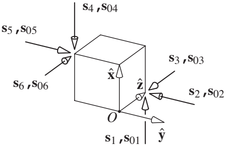
- Let $(\mathbf{t}, \mathbf{t}_0)$ be a differential twist. Reciprocal with respect to $(\mathbf{s}_1, \mathbf{s}_{01})$:

$$t_4 + t_2 = 0$$

Ex 2: Solving the constraint equations

- Reciprocal to all 6 contact screws:

$$\begin{array}{llllllll}
t_4 & & & & +t_2 & & = 0 \\
& -t_5 & & +t_1 & & & = 0 \\
& & -t_6 & & & & = 0 \\
-t_4 & & & & & -t_3 & = 0 \\
& t_5 & & & & +t_3 & = 0 \\
& & t_6 & -t_1 & -t_2 & & = 0
\end{array}$$



- The solutions are of the form

$$(\mathbf{t}, \mathbf{t}_0) = k(1, -1, -1, 1, 1, 0)$$

Ex 2: Interpreting the solution

- The solutions are of the form

$$(\mathbf{t}, \mathbf{t}_0) = k(1, -1, -1, 1, 1, 0)$$

- Pitch: $\mathbf{t} \cdot \mathbf{t}_0 / \mathbf{t} \cdot \mathbf{t} = 0$.
- Point on line closest to origin: $\mathbf{t} \times \mathbf{t}_0 / \mathbf{t} \cdot \mathbf{t} =$
- Direction vector: $\mathbf{t} = (1, -1, -1)$.
- I.e., as expected, the diagonal of the cube.