Lecture 3.
Planar Kinematics

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Mechanics of Manipulation
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Outline

Where are we?

- Kinematics
  1. Foundations and general concepts.
  2. Planar kinematics.
  3. Spherical and spatial kinematics.

Readings etc.

- The text:
  - By now you should have read Chapter 1 of the text. The projective plane is covered in the Appendix of the text.
  - Today's material will take us through Sections 2.1, 2.2, and 2.5.
- Outside:
  - For an interesting history of kinematics, Chapter 1 of Hartenberg and Denavit's *Kinematic Synthesis of Linkages*.
  - Cool linkages etc etc: Reuleaux's *Kinematics of Machinery*.
  - Both the above are openly available at the KMODDL web site `http://kmoddl.library.cornell.edu`.
  - Geometric constructions and linkages simulated on the course web page.
  - Hilbert and Cohn-Vossen. *Geometry and the Imagination*.

Decomposition of displacements

Translation ∘ Rotation

Theorem (2.2)

For any displacement D of $\mathbb{E}^n$, and any point O, D is the composition of a translation with a rotation about O.

Proof.

- Let O' be the image of O under D.
- Let T be the translation taking O to O'.
- Consider the displacement $T^{-1} \circ D$. Where does it map O?

$$(T^{-1} \circ D)(O) = ???$$

- So $T^{-1} \circ D$ is a rotation; call it R.
- So then $T \circ R = T \circ T^{-1} \circ D = D$.

□

Decomposition of displacements

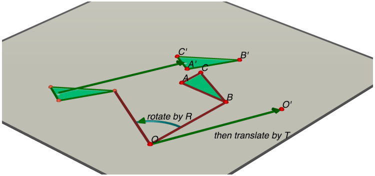
Note:

- Theorem 2.2 is basis for most common representation of displacements.
- The decomposition is not unique: it depends on the choice of O.
- Note how simple it is to prove using group theory. We are dividing the displacement D by the translation T!
- Applies to $\mathbb{E}^n$ for *all* $n \in \mathbb{Z}$.
- Instead of $D = T \circ R$, (rotation, then translation) we could have $D = R \circ U$ (translation, then rotation).
- The order matters. (Planar displacements do not commute!) So $T \neq U$.
- Book has misleading remark in proof of this theorem. See errata file.

Decomposition examples

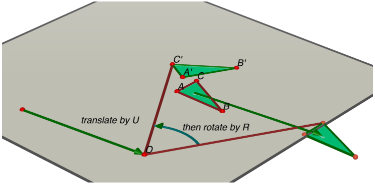
Rotate then translate

$$D = T \circ R$$



Translate then rotate

$$D = R \circ U$$



Planar kinematics

Motivation

That is all we will do on “general” kinematics. On to planar kinematics.

Why planar kinematics?

- The kinematics of flatland? No, much bigger.
- Planar motions are common in $\mathbb{E}^3$. All points moving in parallel planes.
  - Most mobile robots on flat terrain (except when they fall over).
  - Many grippers use planar motion.
  - Many kinematic linkages use planar motion.
- All spatial motions can be decomposed into components including planar motions.
- Spatial rotation is closely related to planar motion.

Two points is enough.

What can we say about rigid motions of $\mathbb{E}^2$?
The first thing is: two points is enough ...

Theorem (2.3)

A planar displacement is completely determined by the motion of any two points.

Proof.

Construct a coordinate frame ... □

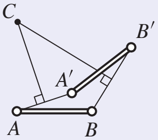
Every D is an R or a T

Theorem (2.4)

Every planar displacement is either a translation or a rotation.

Useful construction

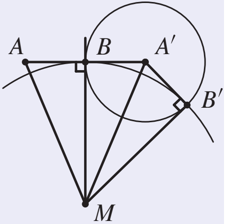
- Pick two points A and B.
- Let A' and B' be the images.
- Construct perpendicular bisectors.
- Intersection gives fixed point. Why? Preserves distance from A and from B.



- Looks like a constructive proof, but it is not.

A legitimate proof

Proof.

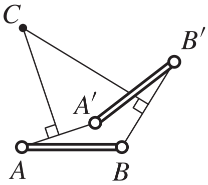


- Pick any point A. We can assume $A \neq A'$.
- Pick B the midpoint of line segment $\overline{AA'}$. We can assume B' is not on $\overline{AA'}$.
- Construct $\perp$ to AB at B, and $\perp$ to $A'B'$ at B'. They are not parallel. Let M be their intersection.
- Consider the rotation R that maps A to A' and M to itself. Where is $R(B)$? Distance constraints give two circles, with two intersections: B or B'.
- So R maps B to B'. $R = D$.

□

Rotation centers

- Consider again construction of rotation centers from the motion of two points. How does it fail when $\overline{AA'}$ is parallel to $\overline{BB'}$?



  - The perpendiculars are parallel. There is no intersection, hence no rotation center.
  - But, in the projective plane they *do* intersect!!!
  - *Every planar displacement is a rotation about a point in the projective plane.*
  - But that is *not* a displacement or a rotation *of* the projective plane. There is no distance, or angle, in the projective plane, hence no rigidity.

Displacements, paths, trajectories.

Definitions

- Recall that a displacement is a rigid change of configuration.

Definition (Trajectory)

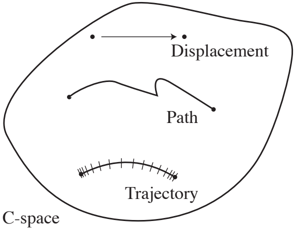
A trajectory $q(t)$ is a continuous function from a time interval to configuration space, i.e. a curve in configuration space parameterized by time.

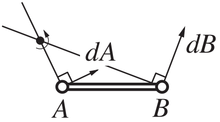
Definition (Path)

A path is a curve $q(s)$ in configuration space parameterized perhaps by arc length.

Constructing rotation centers

- For differentiable trajectory $q(t)$ or a path $q(s)$ we have *velocity* dq/dt or *differential change in configuration* dq.
- To construct instantaneous rotation center for differential displacement, apply limiting process to previous construction.
  - Midpoint of $\overline{AA'} \rightarrow A$.
  - Perpendicular bisector $\rightarrow \perp dA$.
  - Called *velocity center*, or *instantaneous center*, or *IC*.

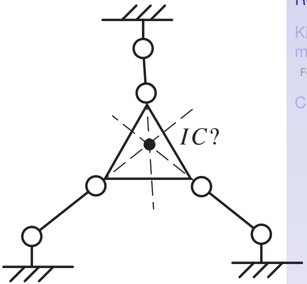




Watch out for false ICs.

To construct IC's:

1. Identify points (e.g. A) and permitted motions (e.g. dA)
2. Construct perpendicular to permitted motion at each point.
3. Intersection of perpendiculars are candidate ICs
   - No intersection means no ICs. It must be immobile.
   - Remember: parallel lines intersect at infinity.



But existence of intersection does not imply mobility!!! Necessary but not sufficient.

Kinematic mechanisms.

Definition (link)

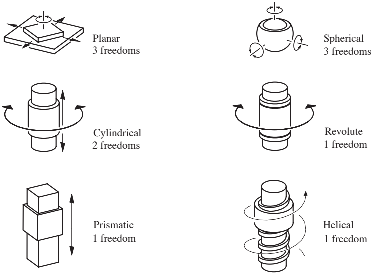
A link is a rigid body.

Definition (joint)

A joint constrains the relative motion of two links.

Definition (lower pair)

A lower pair is a joint involving positive contact area between two links.

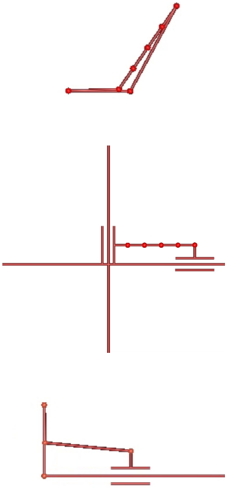


Planar kinematic mechanisms

- Two lower pairs are relevant: revolute and prismatic.

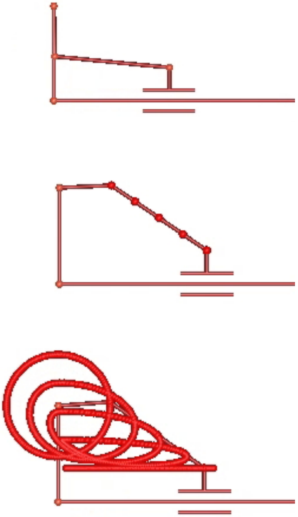
Four bar linkage

- Simplest mobile serial closed chain
- Four links (bars) and four joints.
- Each joint either revolute or prismatic.
- Base link is fixed.
- Two adjacent links are either *cranks* (making circles), *rockers* (making arcs), or *sliders* (prismatic).
- The last link is the *coupler* which can make a variety of complex and useful motions.



Four-bar linkages

- One way to examine coupler motion is to plot locus of several points fixed in the coupler.
- Each coupler curve is a blend of the circle imposed by the crank, and the line imposed by the slider.
- You can use charts of coupler curves to choose linkage for desired function.
- There is a curve for every point in the *moving plane*.

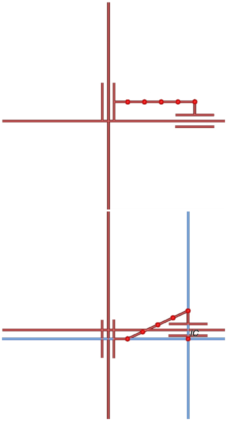


Constructing ICs for four-bar linkages

- Recall the procedure for constructing an IC:

Constructing ICs

- Identify two points *A* and *B*;
- Determine differential motions *dA* and *dB*;
- Construct ⊥ to *dA* at *A*, and ⊥ to *dB* at *B*;
- Intersect to obtain the Instantaneous Center.



Centrodes.

How to characterize planar motion?

1. Instantaneous motion characterized by the *Instantaneous Center*.
2. A more global characterization is given by *coupler curves* (previous slides).
3. Centrodes give an elegant and general global characterization.

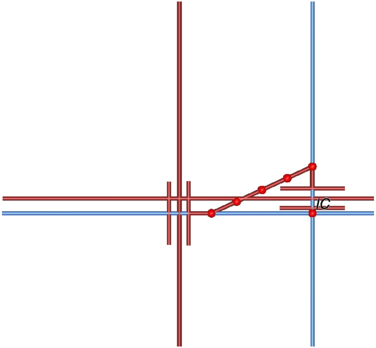
Definition (centrodes)

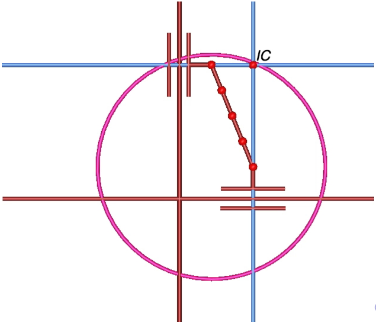
The fixed centrode is the locus of the IC in the fixed plane. The moving centrode is the locus of the IC in the moving plane.

- At any given time, the fixed and moving centrodes must touch at the IC.
- The moving plane rotates about the IC.
- I.e. the moving centrode rolls without slipping on the fixed centrode.

Fixed centrode for elliptic trammels

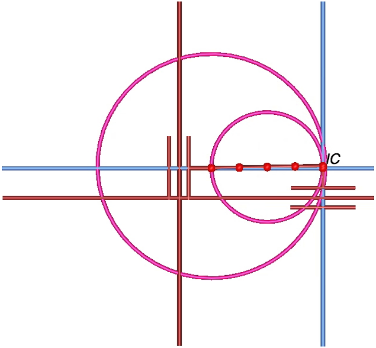
- Use previous technique for constructing IC.
- Repeat for every configuration of linkage, and plot the locus.
- The result is a circle in the fixed plane.
- (Most linkages are not as simple.)





Moving centrode for elliptic trammels

- Use previous technique for constructing IC.
- Repeat for every configuration of linkage, and plot the locus *in the moving plane*.
- (How? Coordinate transform. Or manually using a sheet of acetate or tracing paper.)
- The result is a circle in the moving plane.



- The resulting characterization: the elliptic trammels produce the motion of one circle rolling without slipping on another.
- (Most linkages are not as simple.)

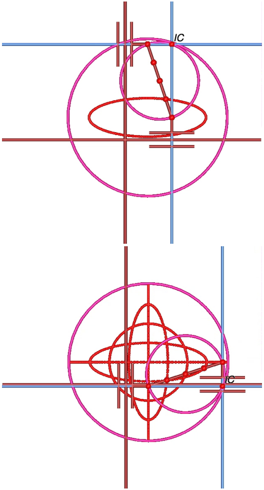
Why "elliptic trammels?"

How to draw an ellipse, 19th century

- Build an elliptic trammel.
- Pick a coupler point on any diameter of the moving centrode.

Other applications

- Simple source of examples and counterexamples.
- Manipulation of block in corner.
- Motion of slipping ladder.
- Motion of mobile robot climbing a step.



Generality of centrodes

- For periodic motions, the centrodes are closed curves.
- For motions over an interval of time, the centrodes are open curves.
- For a sequence of discrete displacements, the centrodes are polygonal curves, called central polygons.
- To construct central polygon for discrete displacements, remember to use the first method, intersecting perpendicular bisectors, to obtain rotation centers.