Spacetime Control
Animation of Natural Phenomena

Goal. Suppose wish to **control** the following physical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

so that it achieves a certain animator goal at a certain time. One way to achieve this is with **gradient-based optimization**.

Step 1. We first **discretize** the system into a sequence of timesteps:

$$\mathbf{x}_0, \mathbf{x}_1, \ldots \mathbf{x}_n$$

using a simulation function which approximates **f**(**x**):

$$\mathbf{x}_{i+1} = \mathbf{S}(\mathbf{x}_i)$$

Step 2. We then **add** control variables, $\mathbf{u}_i$ which allow us to alter the simulation while it's in course. (These are like special "god forces" which we get to add to the system.)

$$\mathbf{x}_{i+1} = \mathbf{S}_i(\mathbf{x}_i, \mathbf{u}_i)$$

We then wish to solve this system across all space-time at once, so we gather the state and control vectors into big vectors:

$$\mathbf{x} = \left[\mathbf{x}_0^T, \mathbf{x}_1^T, \ldots, \mathbf{x}_n^T\right]^T$$
$$\mathbf{u} = \left[\mathbf{u}_0^T, \mathbf{u}_1^T, \ldots, \mathbf{u}_n^T\right]^T$$

Step 3. We express the desired goal as an **objective function**:

$$\phi(\mathbf{x}, \mathbf{u})$$

for which lower values indicate better agreement with the animator goals.

Step 4. The key step is to take **derivatives of this function** with respect to every control variable. By the chain rule, we get:

$$\frac{d\phi}{d\mathbf{u}} = \sum_{i=0}^{n} \frac{\partial\phi}{\partial\mathbf{x}_i}\frac{d\mathbf{x}_i}{d\mathbf{u}} + \frac{\partial\phi}{\partial\mathbf{u}}$$

Expanding the state derivatives we find a recursive structure which mirrors the simulation:

$$\frac{d\mathbf{x}_i}{d\mathbf{u}} = \frac{\partial \mathbf{x}_i}{\partial \mathbf{x}_{i-1}} \frac{d\mathbf{x}_{i-1}}{d\mathbf{u}} + \frac{\partial \mathbf{x}_i}{\partial u}$$

Example

Suppose we wanted to control a 2D ballistic simulation. The state consists of the 2D position and velocity of the object:

$\mathbf{x} = [x, y, \dot{x}, \dot{y}]$

The control parameters $\mathbf{u} = [u_1, u_2]$ control the initial velocity:

$\mathbf{x}_0 = [0, 0, u_1, u_2]$

The state updates as follows:

$\mathbf{x}_{i+1} = \mathbf{x}_i + [\dot{x}_i \Delta t, \dot{y}_i \Delta t, 0, -9.8\Delta t]$

Therefore the derivatives are:

$$\frac{\partial \mathbf{x}_0}{\partial \mathbf{u}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{\partial \mathbf{x}_i}{\partial \mathbf{x}_{i-1}} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we wish to match a particular target position $[\hat{x}, \hat{y}]$ at timestep 9. Then the objective function will be:

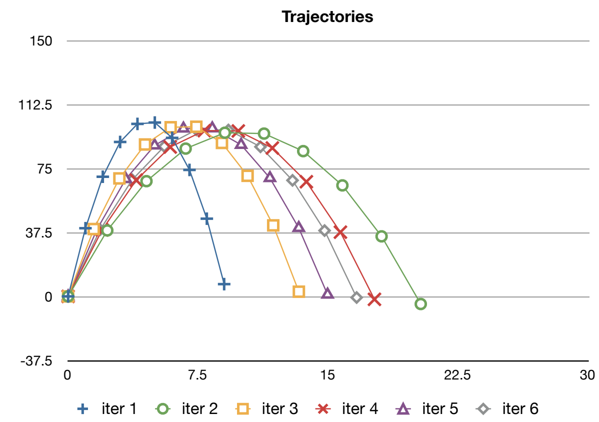
$$\phi(\mathbf{x}) = (\hat{x} - x_9)^2 + (\hat{y} - y_9)^2$$

$$\frac{\partial \phi}{\partial x_9} = 2\left(x_9 - \hat{x}\right)$$

$$\frac{\partial \phi}{\partial y_9} = 2\left(y_9 - \hat{y}\right)$$

Ballistic Motion with Derivative-based Control

Initial Velocity:	1	40								Target:	16	0	
xpos	0	1	2	3	4	5	6	7	8	9	**100.84**		Iteration 1
ypos	0	40	70.2	90.6	101.2	102	93	74.2	45.6	7.2			
xvel	1	1	1	1	1	1	1	1	1	1			
yvel	40	30.2	20.4	10.6	0.8	-9	-18.8	-28.6	-38.4	-48.2			
dxpos/dc1	0	1	2	3	4	5	6	7	8	9	**-126**		
dypos/dc1	0	0	0	0	0	0	0	0	0	0			
dxvel/dc1	1	1	1	1	1	1	1	1	1	1			
dyvel/dc1	0	0	0	0	0	0	0	0	0	0			
dxpos/dc2	0	0	0	0	0	0	0	0	0	0	**129.6**		
dypos/dc2	0	1	2	3	4	5	6	7	8	9			
dxvel/dc2	0	0	0	0	0	0	0	0	0	0			
dyvel/dc2	1	1	1	1	1	1	1	1	1	1			
xpos	0	2.26	4.52	6.78	9.04	11.3	13.56	15.82	18.08	20.34	**38.762896**		Iteration 2
ypos	0	38.704	67.608	86.712	96.016	95.52	85.224	65.128	35.232	-4.464			
xvel	2.26	2.26	2.26	2.26	2.26	2.26	2.26	2.26	2.26	2.26			
yvel	38.704	28.904	19.104	9.304	-0.496	-10.296	-20.096	-29.896	-39.696	-49.496			
dxpos/dc1	0	1	2	3	4	5	6	7	8	9	**78.12**		
dypos/dc1	0	0	0	0	0	0	0	0	0	0			
dxvel/dc1	1	1	1	1	1	1	1	1	1	1			
dyvel/dc1	0	0	0	0	0	0	0	0	0	0			
dxpos/dc2	0	0	0	0	0	0	0	0	0	0	**-80.352**		
dypos/dc2	0	1	2	3	4	5	6	7	8	9			
dxvel/dc2	0	0	0	0	0	0	0	0	0	0			
dyvel/dc2	1	1	1	1	1	1	1	1	1	1			
xpos	0	1.4788	2.9576	4.4364	5.9152	7.394	8.8728	10.3516	11.8304	13.3092	**14.900457222**		Iteration 3
ypos	0	39.50752	69.21504	89.12256	99.23008	99.5376	90.04512	70.75264	41.66016	2.76768			
xvel	1.4788	1.4788	1.4788	1.4788	1.4788	1.4788	1.4788	1.4788	1.4788	1.4788			
yvel	39.50752	29.70752	19.90752	10.10752	0.30752	-9.49248	-19.29248	-29.09248	-38.89248	-48.69248			
dxpos/dc1	0	1	2	3	4	5	6	7	8	9	**-48.4344**		
dypos/dc1	0	0	0	0	0	0	0	0	0	0			
dxvel/dc1	1	1	1	1	1	1	1	1	1	1			
dyvel/dc1	0	0	0	0	0	0	0	0	0	0			
dxpos/dc2	0	0	0	0	0	0	0	0	0	0	**49.81824**		
dypos/dc2	0	1	2	3	4	5	6	7	8	9			
dxvel/dc2	0	0	0	0	0	0	0	0	0	0			
dyvel/dc2	1	1	1	1	1	1	1	1	1	1			
xpos	0	1.963144	3.926288	5.889432	7.852576	9.81572	11.778864	13.742008	15.705152	17.668296	**5.7277357563**		Iteration 4
ypos	0	39.0093376	68.2186752	87.6280128	97.2373504	97.046688	87.0560256	67.2653632	37.6747008	-1.7159616			
xvel	1.963144	1.963144	1.963144	1.963144	1.963144	1.963144	1.963144	1.963144	1.963144	1.963144			
yvel	39.0093376	29.2093376	19.4093376	9.6093376	-0.1906624	-9.9906624	-19.7906624	-29.5906624	-39.3906624	-49.1906624			
dxpos/dc1	0	1	2	3	4	5	6	7	8	9	**30.029328**		
dypos/dc1	0	0	0	0	0	0	0	0	0	0			
dxvel/dc1	1	1	1	1	1	1	1	1	1	1			
dyvel/dc1	0	0	0	0	0	0	0	0	0	0			
dxpos/dc2	0	0	0	0	0	0	0	0	0	0	**-30.8873088**		
dypos/dc2	0	1	2	3	4	5	6	7	8	9			
dxvel/dc2	0	0	0	0	0	0	0	0	0	0			
dyvel/dc2	1	1	1	1	1	1	1	1	1	1			
xpos	0	1.66285072	3.32570144	4.98855216	6.65140288	8.3142536	9.97710432	11.63995504	13.30280576	14.96565648	**2.2017416247**		Iteration 5
ypos	0	39.318210688	68.836421376	88.554632064	98.472842752	98.59105344	88.909264128	69.427474816	40.145685504	1.063896192			
xvel	1.66285072	1.66285072	1.66285072	1.66285072	1.66285072	1.66285072	1.66285072	1.66285072	1.66285072	1.66285072			
yvel	39.318210688	29.518210688	19.718210688	9.918210688	0.118210688	-9.681789312	-19.48178931	-29.28178931	-39.08178931	-48.88178931			
dxpos/dc1	0	1	2	3	4	5	6	7	8	9	**-18.61818336**		
dypos/dc1	0	0	0	0	0	0	0	0	0	0			
dxvel/dc1	1	1	1	1	1	1	1	1	1	1			
dyvel/dc1	0	0	0	0	0	0	0	0	0	0			
dxpos/dc2	0	0	0	0	0	0	0	0	0	0	**19.150131456**		
dypos/dc2	0	1	2	3	4	5	6	7	8	9			
dxvel/dc2	0	0	0	0	0	0	0	0	0	0			
dyvel/dc2	1	1	1	1	1	1	1	1	1	1			
xpos	0	1.8490325536	3.6980651072	5.5470976608	7.3961302144	9.245162768	11.094195322	12.943227875	14.792260429	16.641292982	**0.8463494805**		Iteration 6
ypos	0	39.126709373	68.453418747	87.98012812	97.706837494	97.633546867	87.760256241	68.086965614	38.613674988	-0.659615639			
xvel	1.8490325536	1.8490325536	1.8490325536	1.8490325536	1.8490325536	1.8490325536	1.8490325536	1.8490325536	1.8490325536	1.8490325536			
yvel	39.126709373	29.326709373	19.526709373	9.7267093734	-0.073290627	-9.873290627	-19.67329063	-29.47329063	-39.27329063	-49.07329063			
dxpos/dc1	0	1	2	3	4	5	6	7	8	9	**11.543273683**		
dypos/dc1	0	0	0	0	0	0	0	0	0	0			
dxvel/dc1	1	1	1	1	1	1	1	1	1	1			
dyvel/dc1	0	0	0	0	0	0	0	0	0	0			
dxpos/dc2	0	0	0	0	0	0	0	0	0	0	**-11.8730815**		
dypos/dc2	0	1	2	3	4	5	6	7	8	9			
dxvel/dc2	0	0	0	0	0	0	0	0	0	0			
dyvel/dc2	1	1	1	1	1	1	1	1	1	1			



In this example, we controlled a object flying through the air under the influence of gravity. Each iteration, we measure derivatives and update the initial velocity opposite the direction of the gradient.