Deformable Materials 2

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source: Müller, Stam, James, Thürey. Real-Time Physics Class Notes.
Goal
Overview

- Strain (Recap)
- Stress
- From Strain to Stress
- Discretization
- Simulation
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Deformations

Spring deformed by $\Delta x$:

$$\text{force} = -E \Delta x$$

stress: $\sigma$

strain: $\epsilon$

Young’s modulus

Hooke’s Law:

$$\sigma = E \epsilon$$
In 3D...

\[ u(x) = p(x) - x \]
Defining Strain

- Strain is **invariant to translation**.
  - Ignore $\rho(x)$
  - Define in terms of local coordinate system transform: $\nabla \rho(x)$.

- Strain is **invariant to rotation**.
  - If $[\nabla \rho(x)]^T \nabla \rho(x) = I$,
  - Then $\varepsilon = 0$

- Natural to define strain as:
  - $\varepsilon = \frac{1}{2} ([\nabla \rho(x)]^T \nabla \rho(x) - I)$
  - 6 DOFs
Green’s Strain

\[ \varepsilon_G = \frac{1}{2} \left( \nabla u + \left[ \nabla u \right]^T + [\nabla u]^T \nabla u \right) \]
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Stress

\[ \sigma = E \varepsilon \]
Stress

Direct Stress:

Direct stresses cause compression.

\( \sigma_{xx}, \sigma_{yy}, \sigma_{zz} \)

Shear Stress:

Shear stresses resist compression.

\( \sigma_{xy}, \sigma_{yz}, \sigma_{xz} \)

Stress Tensor:

\[
\sigma = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{xz} & \sigma_{yz} & \sigma_{zz}
\end{bmatrix}
\]
4.1.3 Stress
capture the rotation correctly. This is an important observation we will discuss in Sec.

In this case, only Green’s non-linear tensor yields the correct result; its linearization cannot

deformations. In three dimensions, Hooke’s law reads

\[
\sigma = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
\]

The two measures differ but only by a constant scalar factor. With the following interpretation: As we saw before, at a single material point, the strain

\[
\varepsilon = \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{xy} \\
\varepsilon_{xz}
\end{bmatrix}
\]

The stress tensor is multiplied by

\[
\sigma = \mathbf{C} \varepsilon
\]

\( \mathbf{C} \) is the stiffness tensor, which relates the stress and strain. The two main coefficients are

\[
\mathbf{C} = \begin{bmatrix}
E & 0 & 0 \\
0 & E & 0 \\
0 & 0 & \nu E
\end{bmatrix}
\]

Now let us turn to the measurement of stress: the force per unit area. As strain, stress is

\[
\sigma_{xy} \cdot \mathbf{n}
\]

Stress measures the force on each face:

\[
\frac{df}{dA} = \sigma \cdot \mathbf{n},
\]

where

\[
\mathbf{n} = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

Even though the displacement field should not generate any strain, we have

\[
\varepsilon = \frac{1}{2} \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T \right)
\]

\( \nabla \mathbf{u} \) is the displacement gradient. The two strain measures differ but only by a constant scalar factor.
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Young’s Modulus

\[ \sigma = E \epsilon \]
Voigt Notation

Stress

\[ \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \]

\{\sigma\} = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}]^T \in \mathbb{R}^6

Strain

\[ \varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \]

\{\varepsilon\} = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{xz}]^T \in \mathbb{R}^6
Isotropic Materials

\[ \{ \sigma \} = E \{ \epsilon \} \quad E \in \mathbb{R}^{6 \times 6} \]

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy} \\
\sigma_{yz} \\
\sigma_{zx}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & 1-2\nu & 0 & 0 \\
0 & 0 & 0 & 0 & 1-2\nu & 0 \\
0 & 0 & 0 & 0 & 0 & 1-2\nu
\end{bmatrix} \begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
\epsilon_{zz} \\
\epsilon_{xy} \\
\epsilon_{yz} \\
\epsilon_{zx}
\end{bmatrix}
\]

**Elastic Stiffness**

How strongly the material resists deformation.

**Poisson’s Ratio**

How much volume is conserved.

\( \nu \in \left[ 0, \frac{1}{2} \right] \)
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Discretization

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Discrete Strain

\[ p(x) = [p_1, p_2, p_3] [x_1, x_2, x_3]^{-1} x = Px \]

\[ \nabla p = P \text{ and } \nabla u = P - I \]

\[ \varepsilon = \frac{1}{2} (\nabla u + [\nabla u]^T + [\nabla u]^T \nabla u) \]
Simulation Loop

- Compute Strain:
  \[ \varepsilon = \frac{1}{2} (\nabla \mathbf{u} + [\nabla \mathbf{u}]^T + [\nabla \mathbf{u}]^T \nabla \mathbf{u}) \]

- Convert to Stress:
  \[ \sigma = E \varepsilon \]
  \[ \frac{d \mathbf{f}}{dA} = \sigma \cdot \mathbf{n}. \]

- Compute Face Forces:
  \[ \mathbf{f}_{0,1,2} = \sigma \cdot \mathbf{n}_{0,1,2} \cdot A_{0,1,2} \]
  \[ = \sigma \left[ (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0) \right] \]

- Distribute to vertices.

- Integrate eqns of motion (e.g. 4th order RK).
Examples
Question...

• Why does the strain matrix have only 6 degrees of freedom?