

# Rigid Body Dynamics

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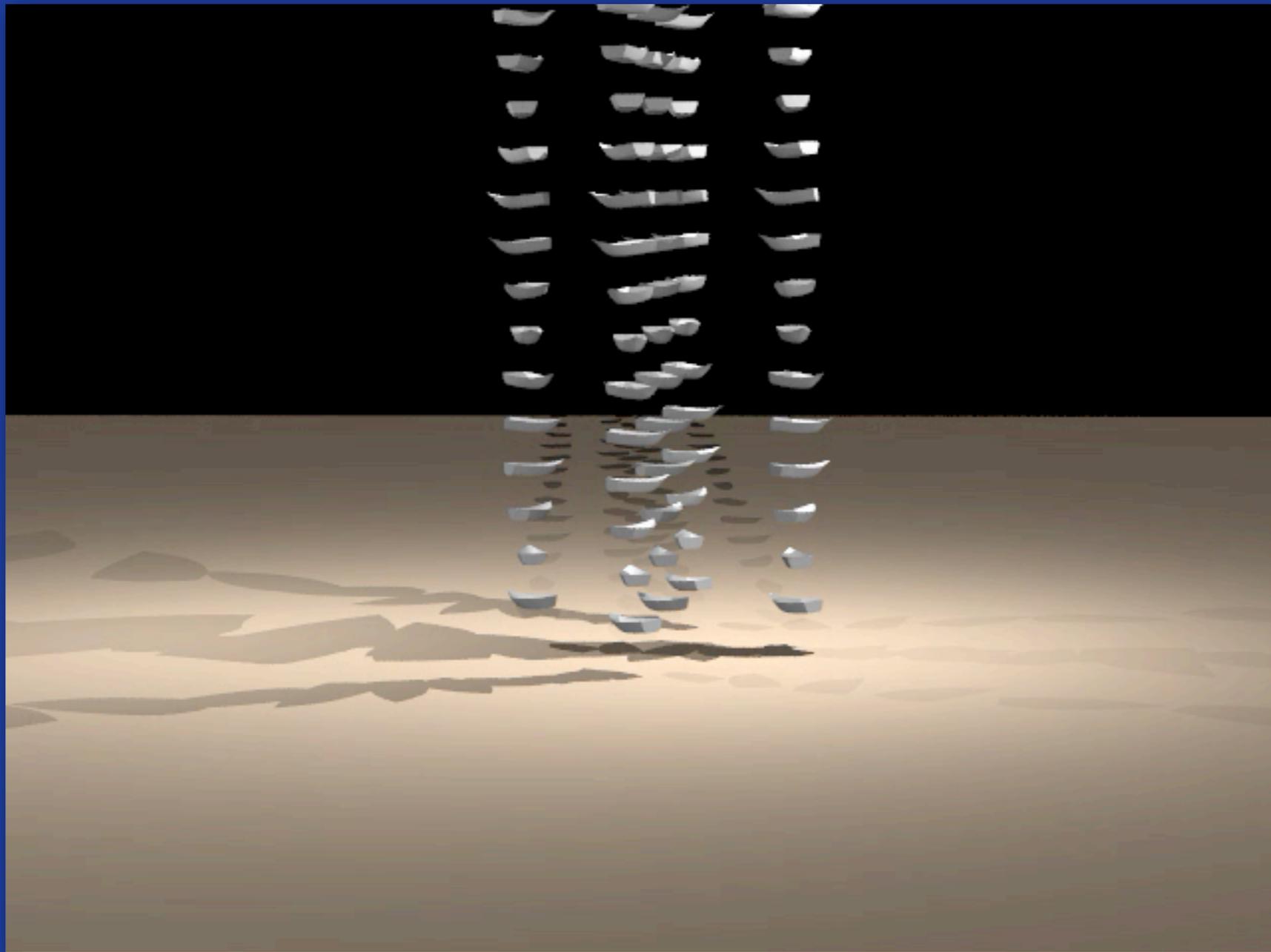
# Rigid Body Dynamics

## Rigid Body Dynamics

*David Baraff*

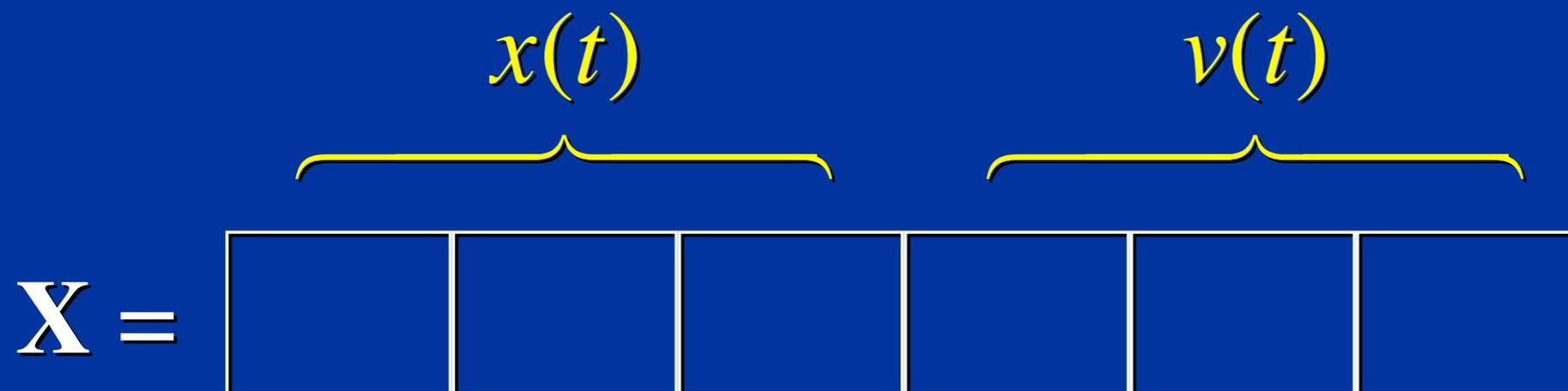


# What is a Rigid Body?



# Particle State

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$

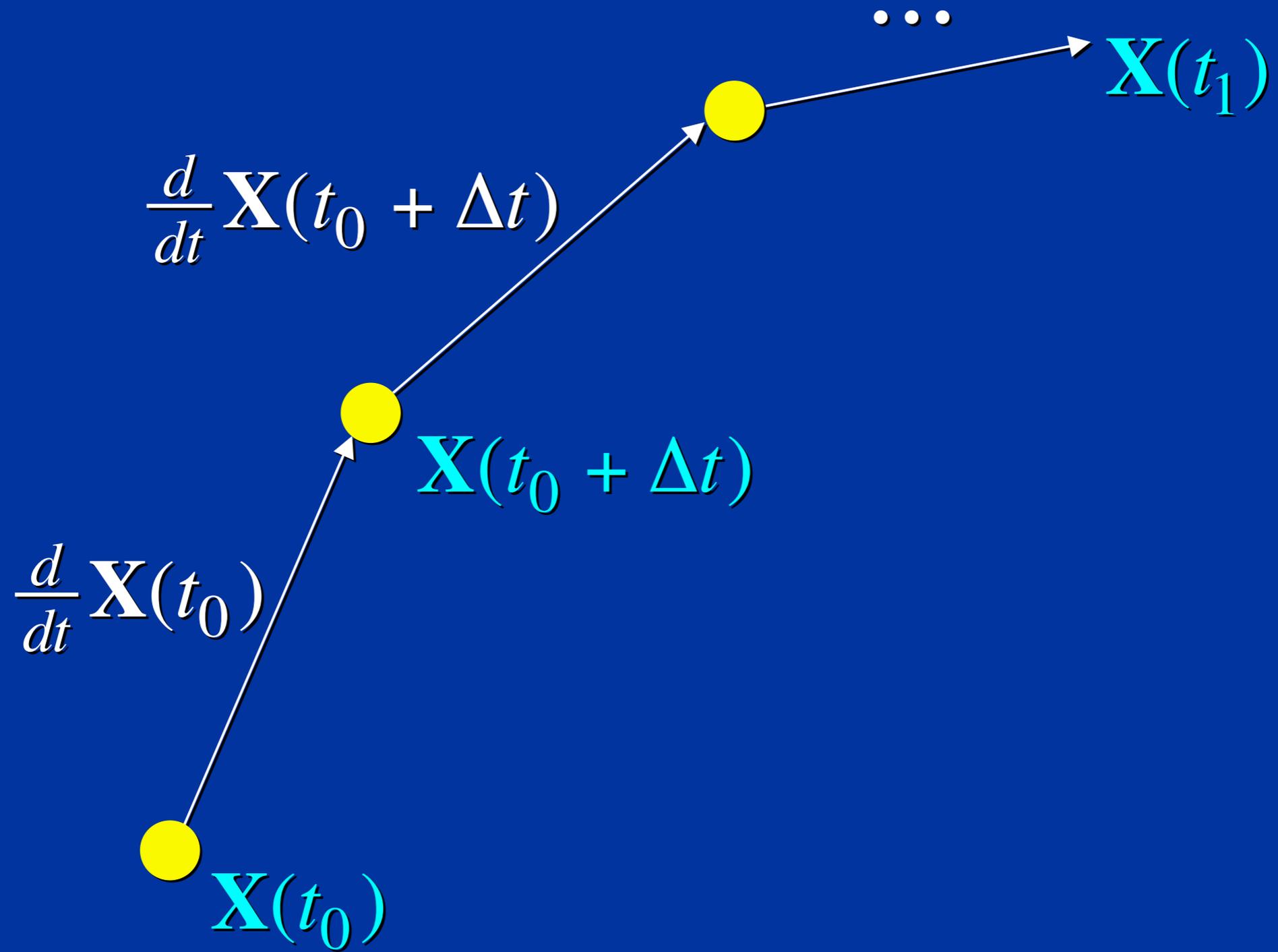


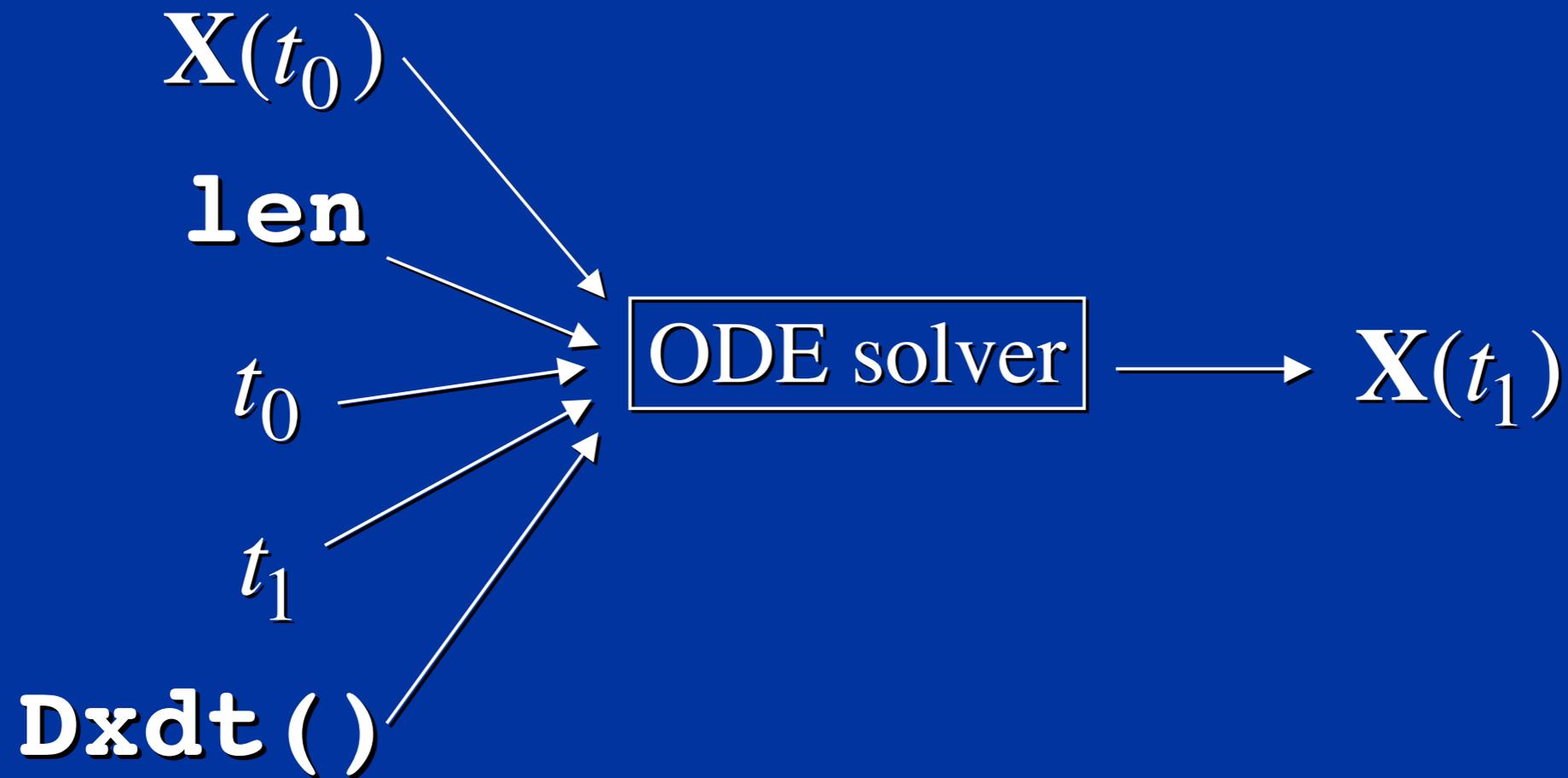
# State Derivative

$$\frac{d}{dt} \mathbf{X} = \frac{d}{dt} \begin{pmatrix} x_1(t) \\ v_1(t) \\ \vdots \\ x_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ F_1(t)/m_1 \\ \vdots \\ v_n(t) \\ F_n(t)/m_n \end{pmatrix}$$

$$\frac{d}{dt} \mathbf{X} = \begin{table border="1" style="display: inline-table; border-collapse: collapse;">|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | ... 6n elements ... |  |  |$$

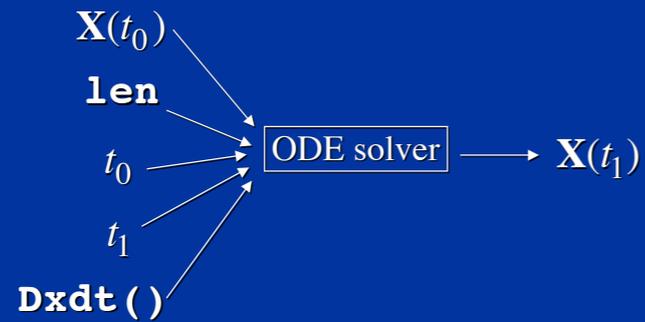
# ODE solution





```
void Dxdt(double t, double x[],  
          double xdot[])
```

# What We Have



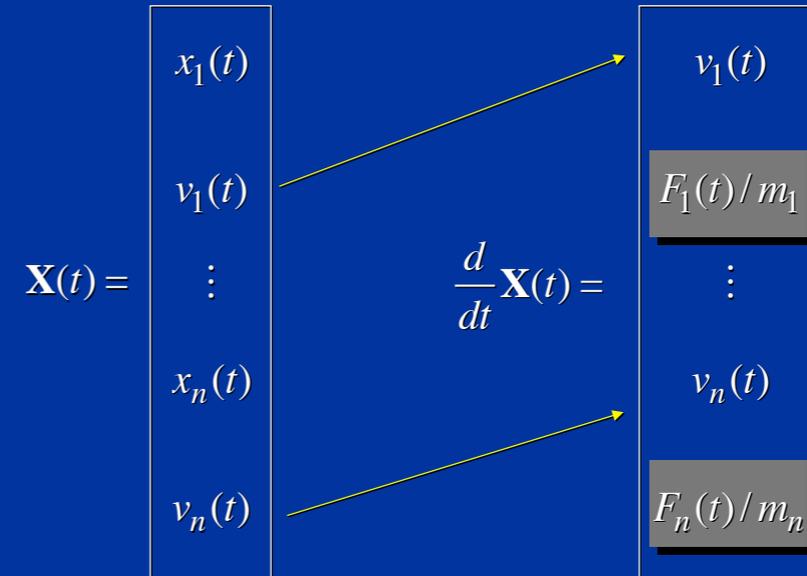
```
void Dxdt(double t, double x[],  
          double xdot[])
```

SIGGRAPH 2001 COURSE NOTES

SF9

PHYSICALLY BASED MODELING

**Dxdt()**

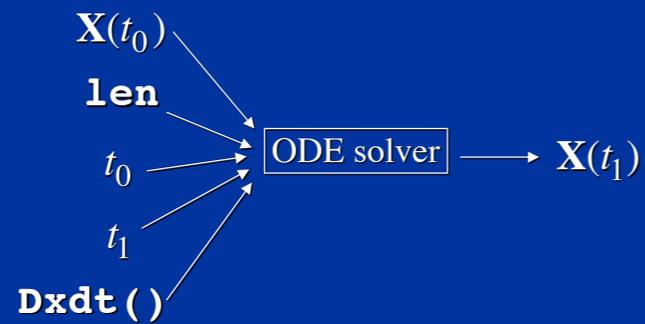


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PHYSICALLY BASED MODELING

# Our Goal



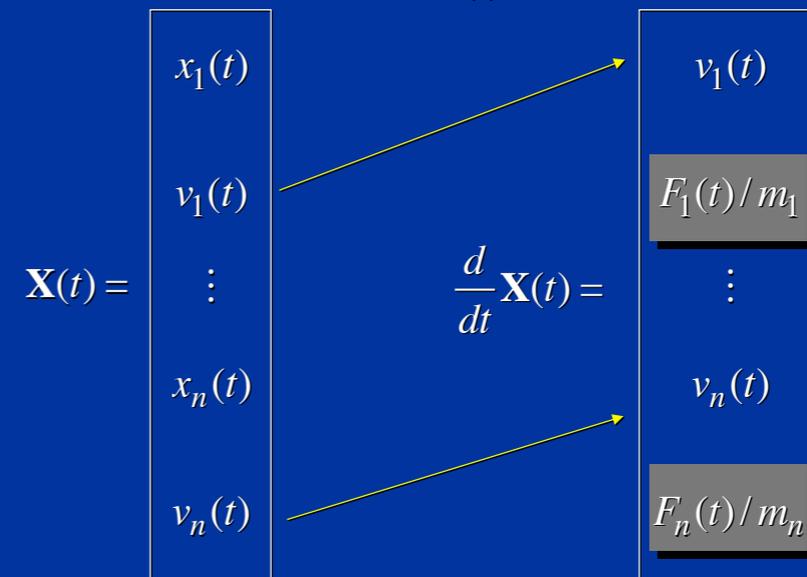
```
void Dxdt(double t, double x[],  
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PHYSICALLY BASED MODELING

**Dxdt()**



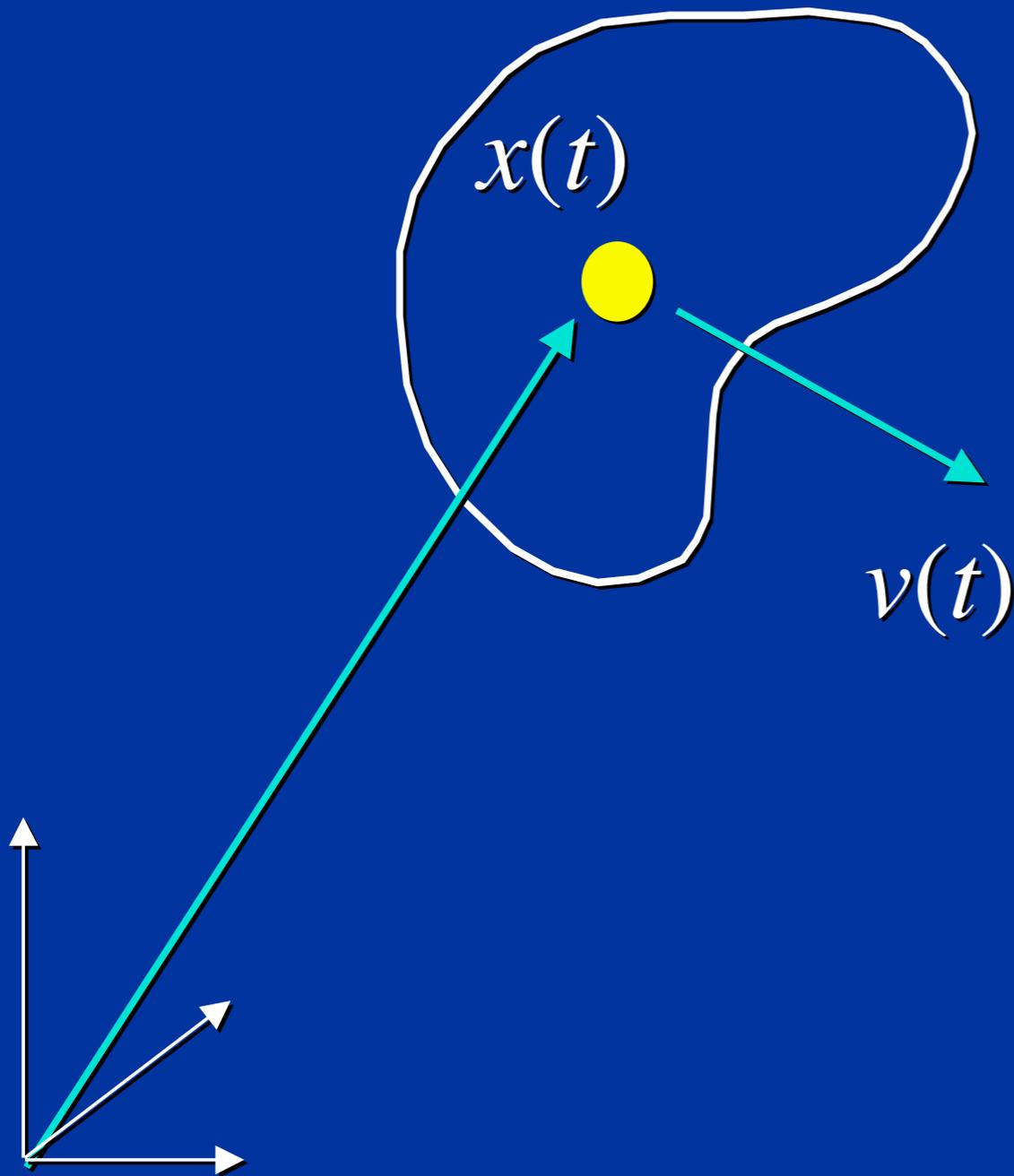
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PHYSICALLY BASED MODELING

Replicate this approach  
for rigid bodies.

# Rigid Body State



$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ ? \\ v(t) \\ ? \end{pmatrix}$$

# Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{X}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ \text{?} \\ Mv(t) \\ \text{?} \end{pmatrix} = \begin{pmatrix} \text{?} \end{pmatrix}$$

- Use Momentum  $P=Mv$  instead of just  $v$ .
- What is this?

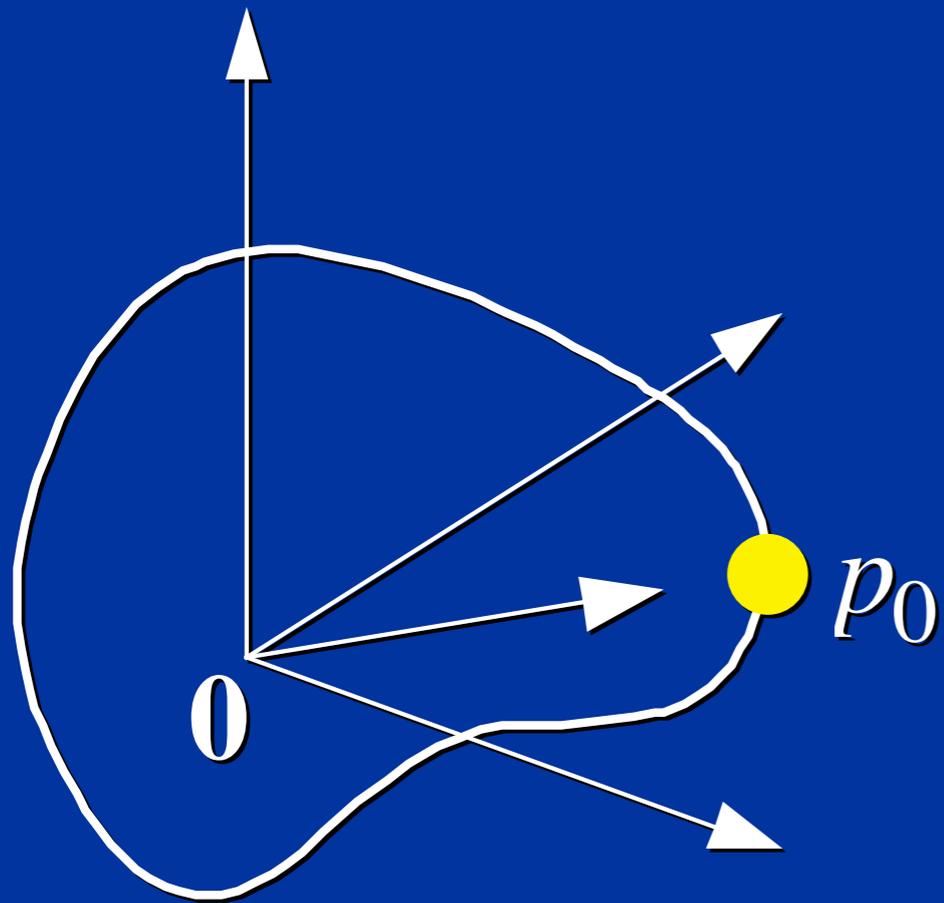
# Orientation

We represent orientation as a rotation matrix<sup>†</sup>  $\mathbf{R}(t)$ . Points are transformed from body-space to world-space as:

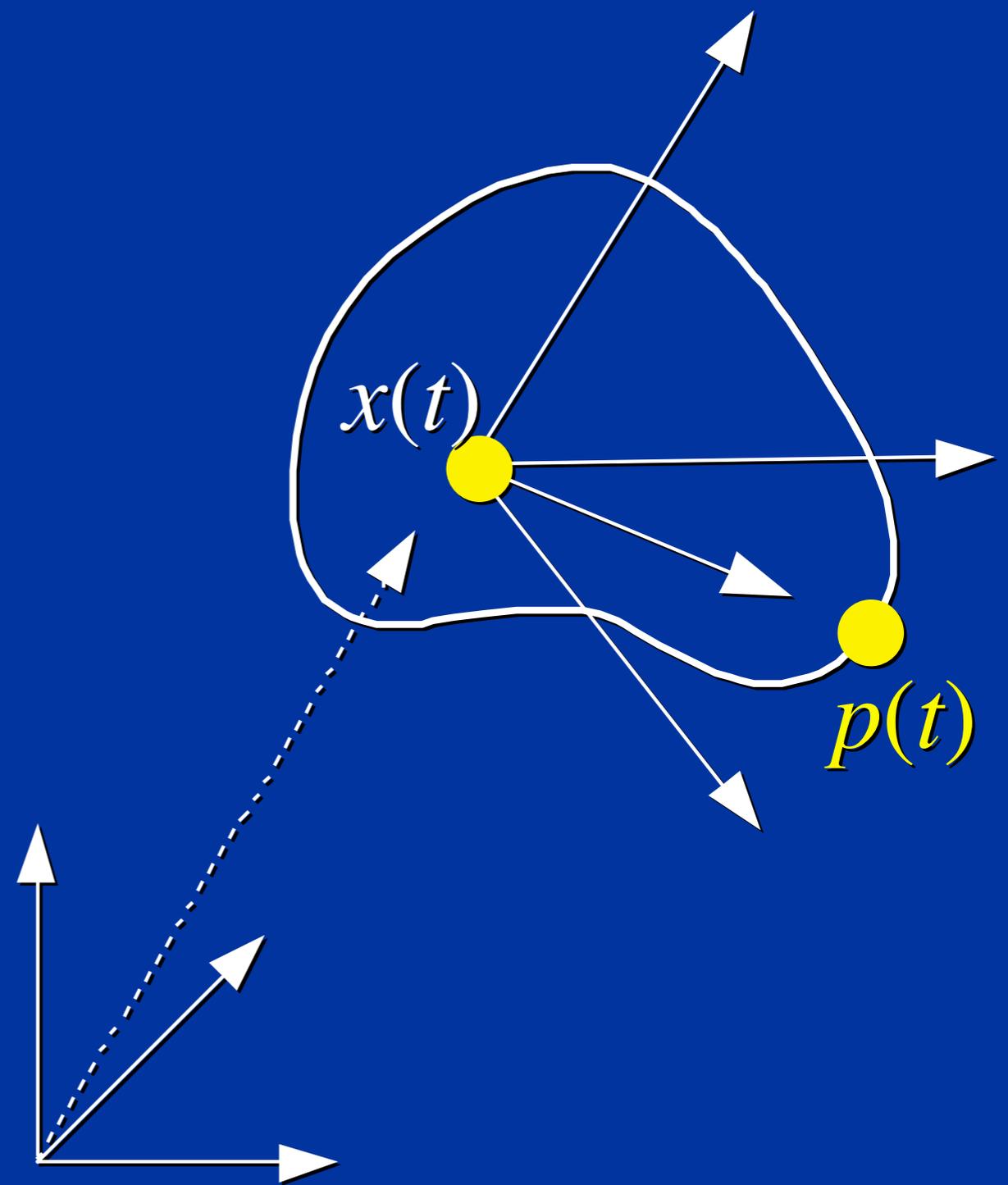
$$p(t) = \mathbf{R}(t)p_0 + x(t)$$

---

<sup>†</sup>He's lying. Actually, we use quaternions.



body space



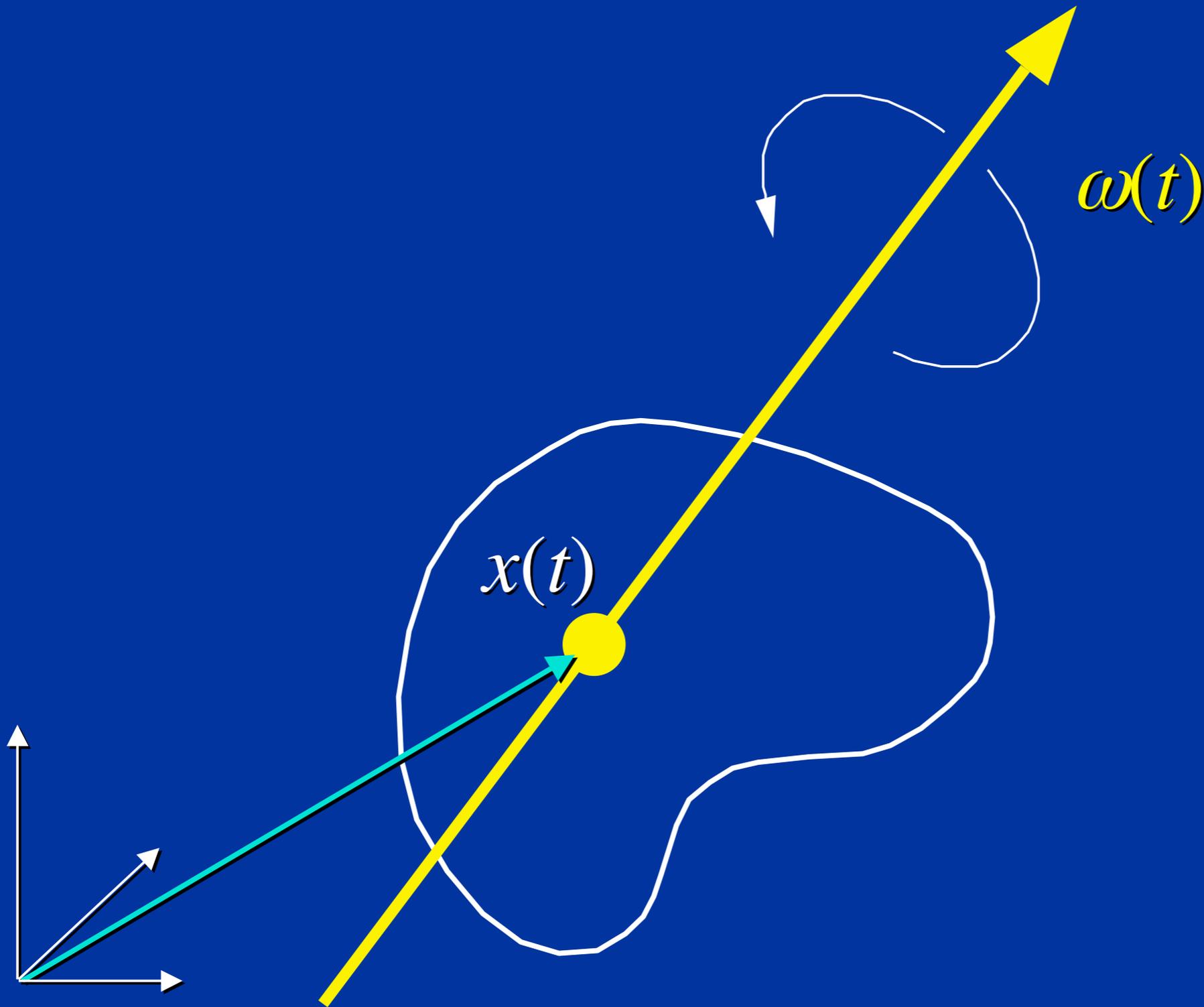
world space

# Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{X}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \text{?} \end{pmatrix} = \begin{pmatrix} \text{?} \end{pmatrix}$$

- What is this?

# Angular Velocity Definition

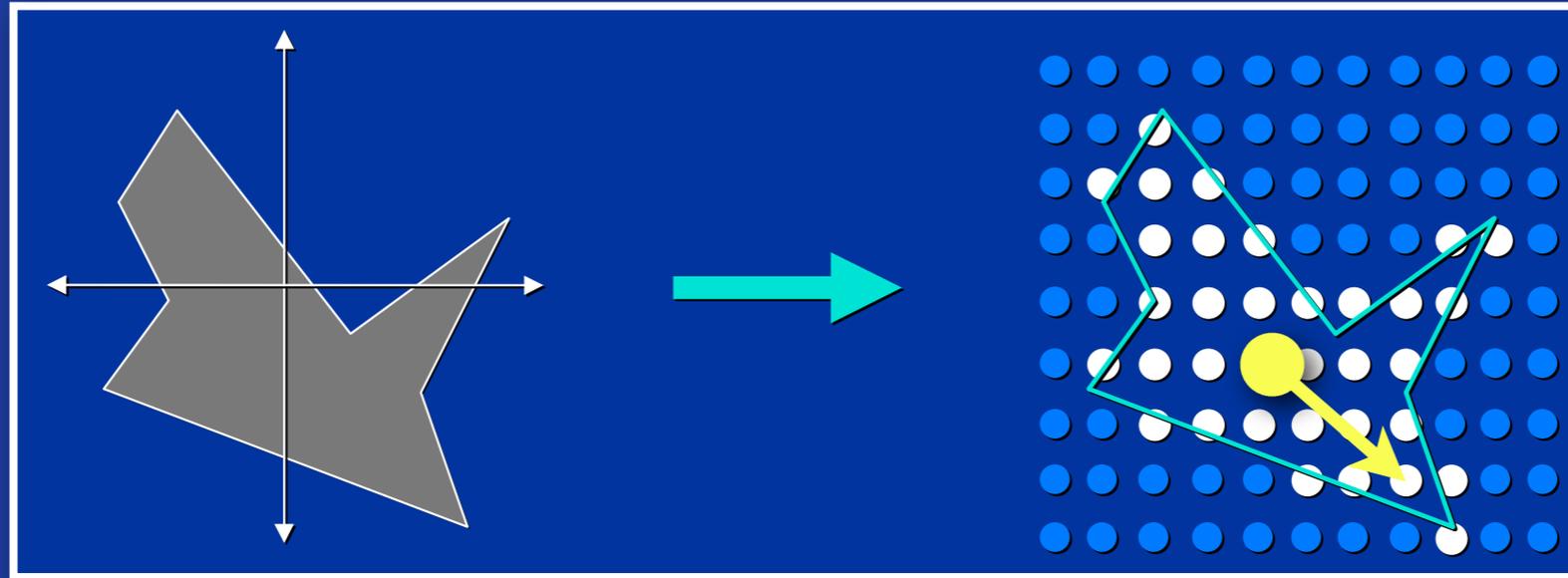


# Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{X}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \omega(t) \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

- What is this?

# Discretized View



- **Total Mass:**

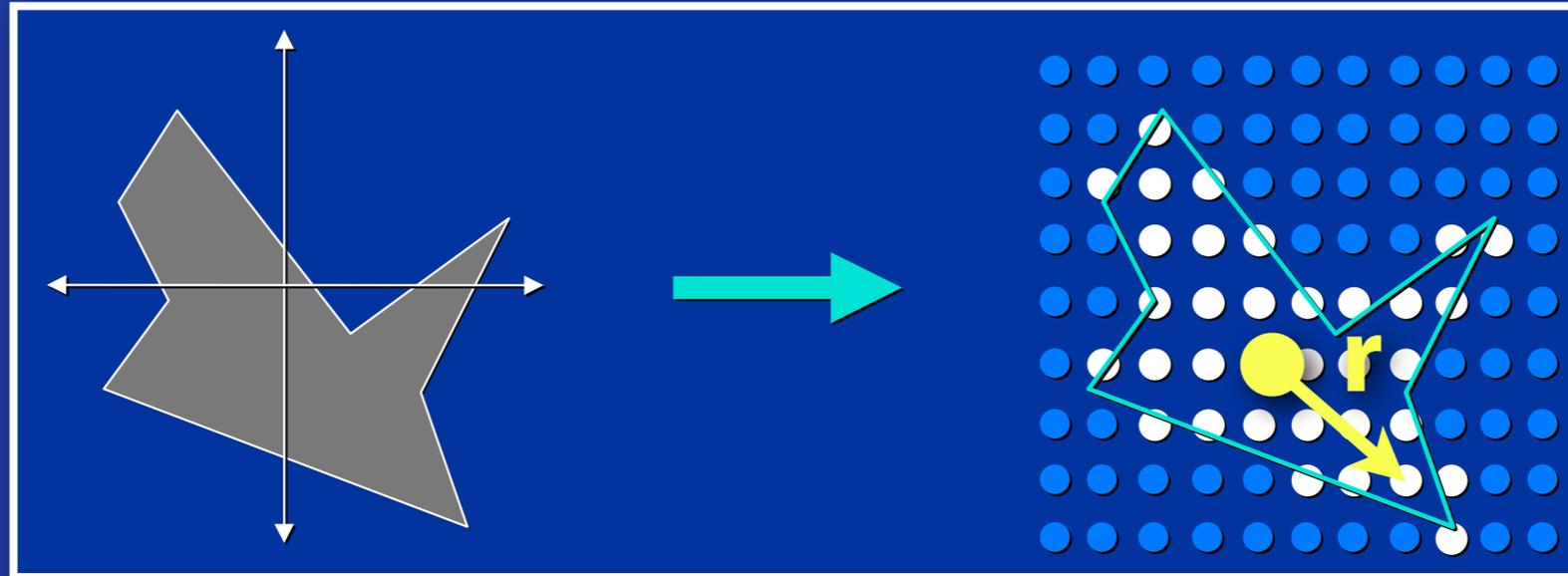
$$M = \sum_i m_i$$

- **Center of Mass:**

$$\bar{\mathbf{x}} = \frac{1}{M} \sum_i m_i \mathbf{x}_i$$

- **Relative Position:**  $\mathbf{r}_i = \mathbf{x}_i - \bar{\mathbf{x}}$

# Discretized View



- **Basic Principles:**

- Conservation of **Linear Momentum**

$$\frac{d}{dt} \sum_i m_i \dot{\mathbf{x}}_i = 0$$

- Conservation of **Angular Momentum**

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = 0$$

# Conservation and Forces

## Linear Momentum

$$\frac{d}{dt} \sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F}$$

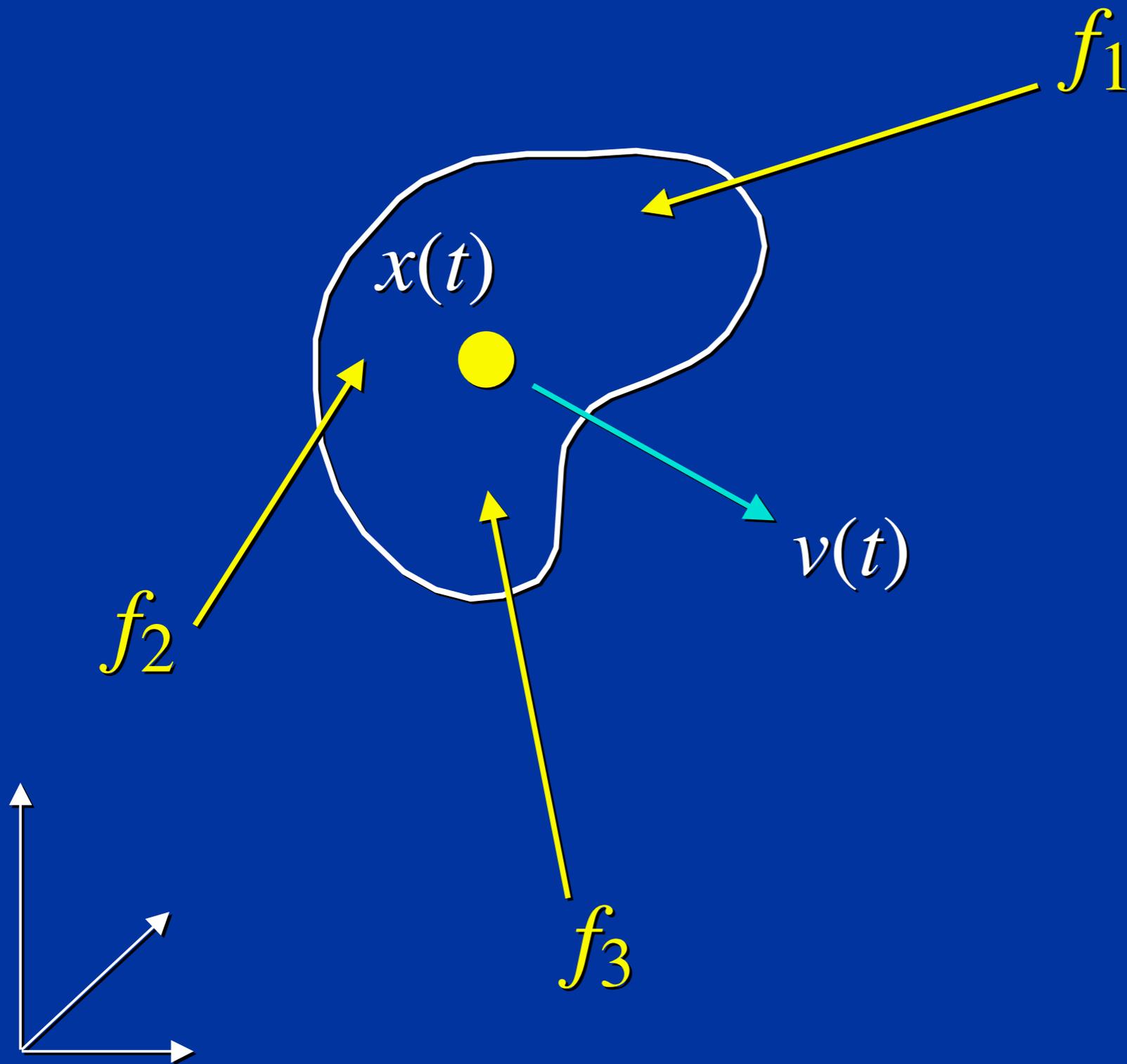
|| (def)

$$\left( \bar{\mathbf{x}} = \frac{1}{M} \sum_i m_i \mathbf{x}_i \right)$$

$$\left( M \ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i \right)$$

$$M \ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

# Net Force



$$F(t) = \sum f_i$$

# Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{X}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \omega(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ ? \\ F(t) \\ ? \end{pmatrix}$$

- What are these?

# Angular Velocity

We represent angular velocity as a vector  $\omega(t)$ , which encodes both the axis of the spin and the speed of the spin.

How are  $\mathbf{R}(t)$  and  $\omega(t)$  related?

# Angular Velocity

•  $\dot{\mathbf{R}}(t)$  and  $\omega(t)$  are related by:

$$\frac{d}{dt}\mathbf{R}(t) = \begin{pmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{pmatrix} \mathbf{R}(t)$$
$$= \omega(t)^* \mathbf{R}(t)$$

$\omega^*$  can be viewed as the matrix form of  $-(\omega \times)$

# Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{X}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ \mathbf{R}(t) \\ Mv(t) \\ \langle \omega(t) \rangle \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* \mathbf{R}(t) \\ F(t) \\ ? \end{pmatrix}$$

Need to relate  $\dot{\omega}(t)$  and mass distribution to  $F(t)$ .

# Conservation and Forces

## Linear Momentum

$$\frac{d}{dt} \sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F} \quad \text{|| (def)}$$

$$\left( \bar{\mathbf{x}} = \frac{1}{M} \sum_i m_i \mathbf{x}_i \right)$$

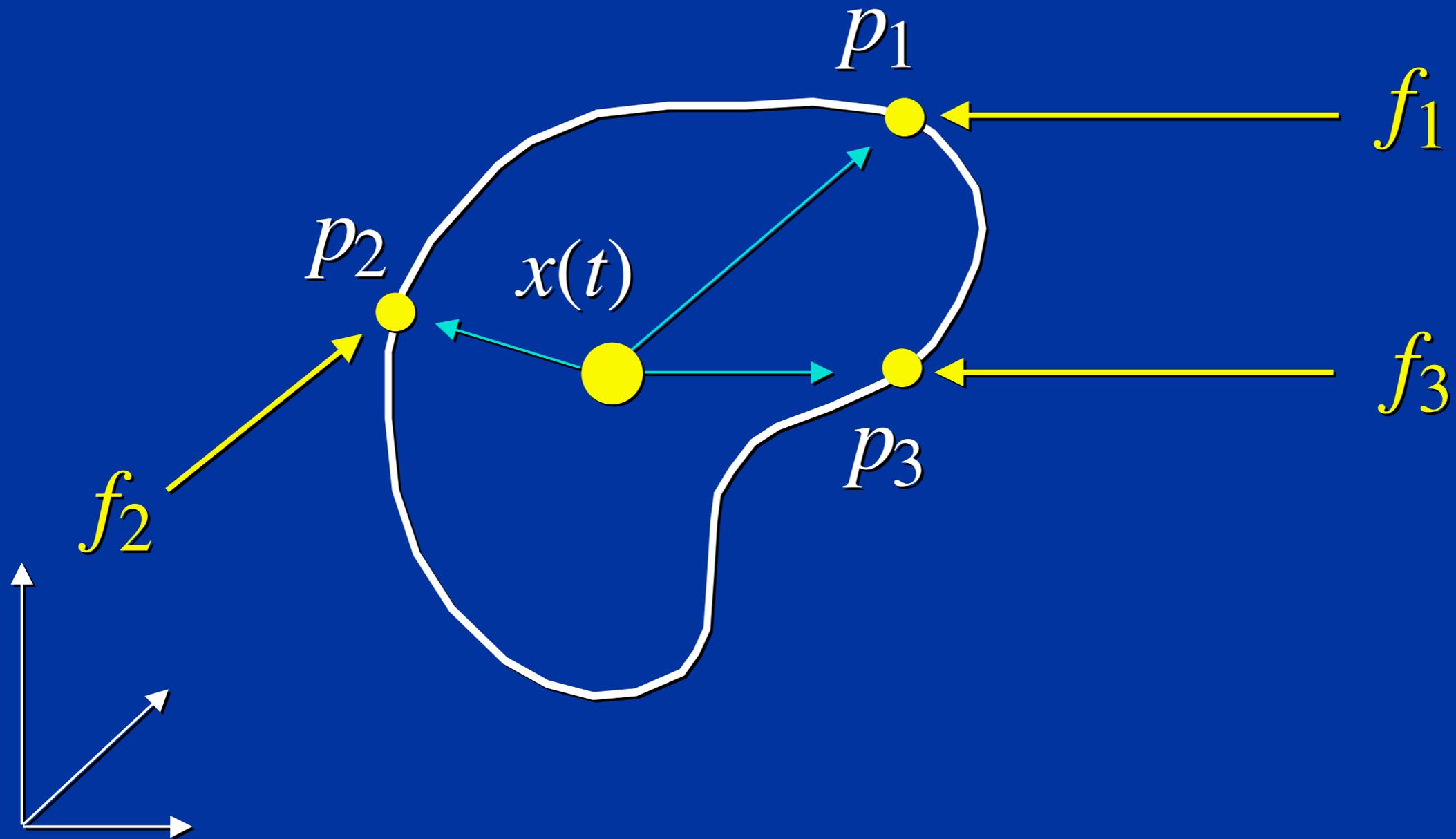
$$\left( M \ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i \right)$$

$$M \ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

## Angular Momentum

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

# Net Torque



$$\tau(t) = \sum (p_i - x(t)) \times f_i$$

# Conservation and Forces

## Linear Momentum

$$\frac{d}{dt} \sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F} \quad \text{|| (def)}$$

$$\left( \bar{\mathbf{x}} = \frac{1}{M} \sum_i m_i \mathbf{x}_i \right)$$

$$\left( M \ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i \right)$$

$$M \ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

## Angular Momentum

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \tau$$

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \boldsymbol{\omega} \times \mathbf{r}_i = \tau$$

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^* \boldsymbol{\omega} = \tau$$

# Discrete Inertia

$$I = \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^*$$

$$I = \sum_i \left( m_i \begin{bmatrix} -y^2 - z^2 & xy & xz \\ xy & -x^2 - z^2 & yz \\ xz & yz & -x^2 - y^2 \end{bmatrix} \right)$$

# Conservation and Forces

## Linear Momentum

$$\frac{d}{dt} \sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F} \quad \text{|| (def)}$$

$$\left( \bar{\mathbf{x}} = \frac{1}{M} \sum_i m_i \mathbf{x}_i \right)$$

$$\left( M \ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i \right)$$

$$M \ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

## Angular Momentum

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \tau$$

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \boldsymbol{\omega} \times \mathbf{r}_i = \tau$$

$$\frac{d}{dt} \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^* \boldsymbol{\omega} = \tau$$

$$\frac{d}{dt} I \boldsymbol{\omega} = \tau$$

# Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{X}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ \mathbf{R}(t) \\ M\mathbf{v}(t) \\ \mathbf{I}(t)\boldsymbol{\omega}(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \boldsymbol{\omega}(t)^* \mathbf{R}(t) \\ F(t) \\ \boldsymbol{\tau}(t) \end{pmatrix}$$

$P(t)$  – linear momentum

$L(t)$  – angular momentum

# Discrete Inertia

$$I = \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^*$$

$$I = \sum_i \left( m_i \begin{bmatrix} -y^2 - z^2 & xy & xz \\ xy & -x^2 - z^2 & yz \\ xz & yz & -x^2 - y^2 \end{bmatrix} \right)$$

# Continuous Inertia

$$\mathbf{I}(t) = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

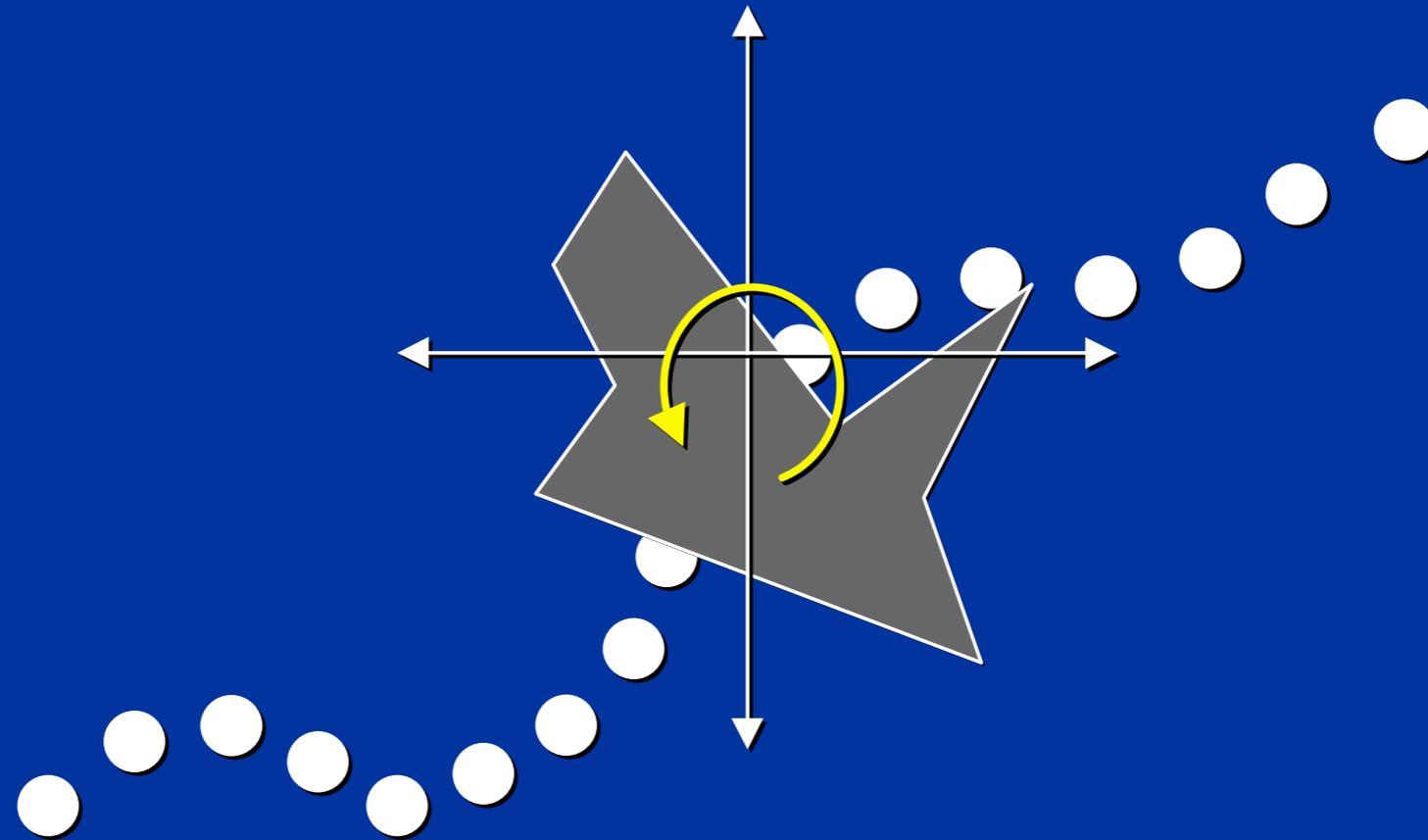
diagonal terms

off-diagonal terms

$$I_{xx} = M \int_V (y^2 + z^2) dV$$

$$I_{xy} = -M \int_V xy dV$$

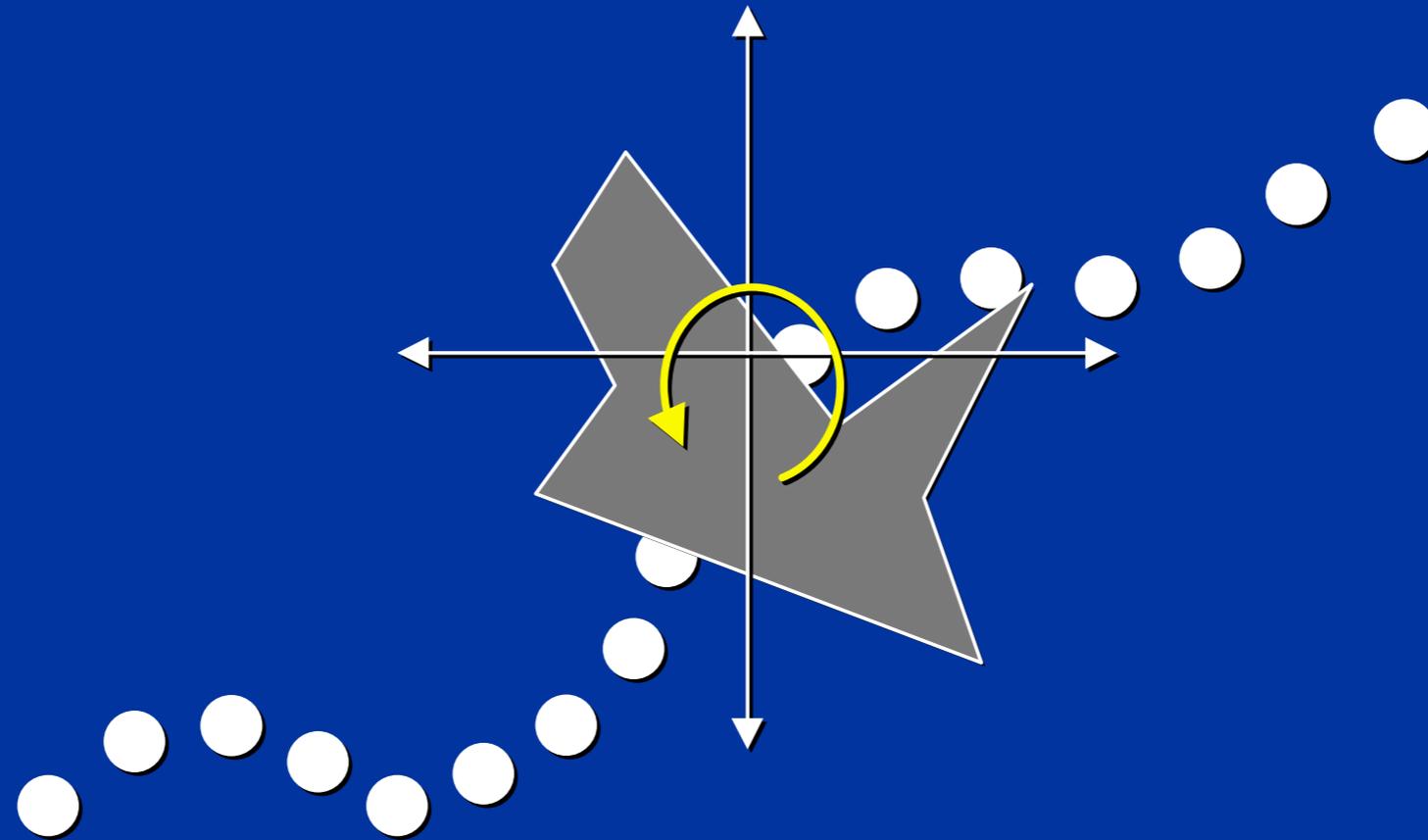
# Inertia Tensors Vary in World Space...



$$I_{xx} = M \int_V (y^2 + z^2) dV$$

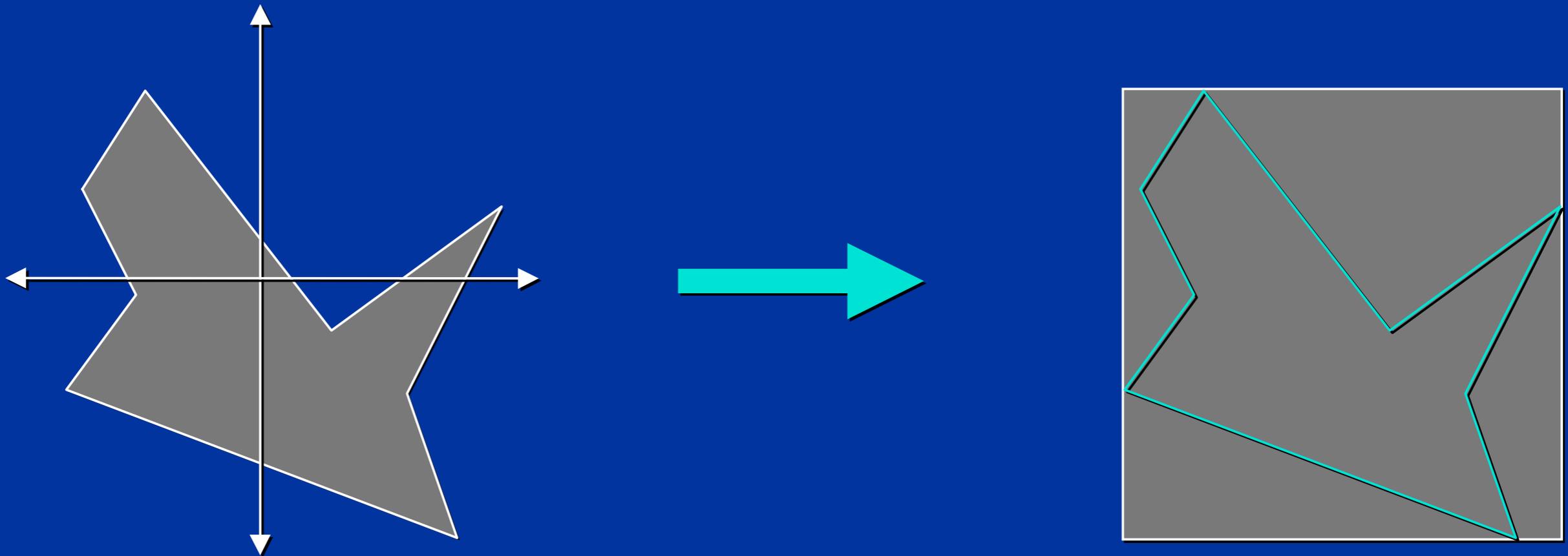
$$I_{xy} = -M \int_V xy dV$$

... but are **Constant in Body Space**



$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{\text{body}}\mathbf{R}(t)^T$$

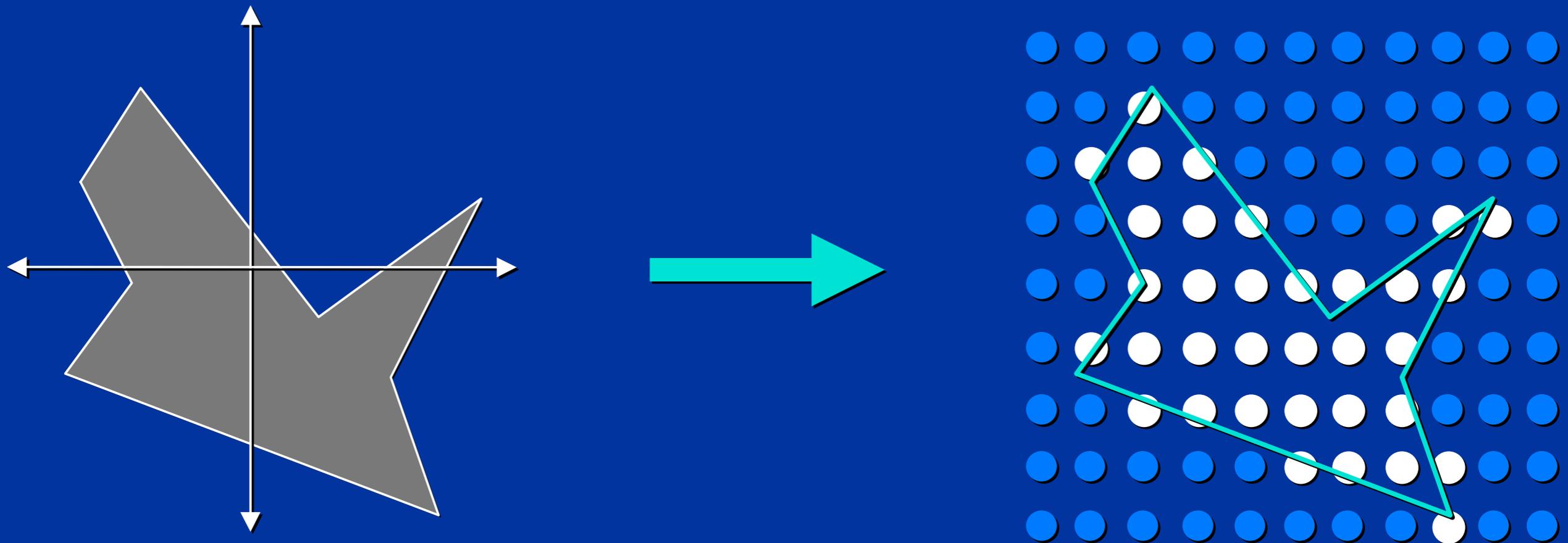
# Approximating $I_{\text{body}}$ : Bounding Boxes



Pros: Simple.

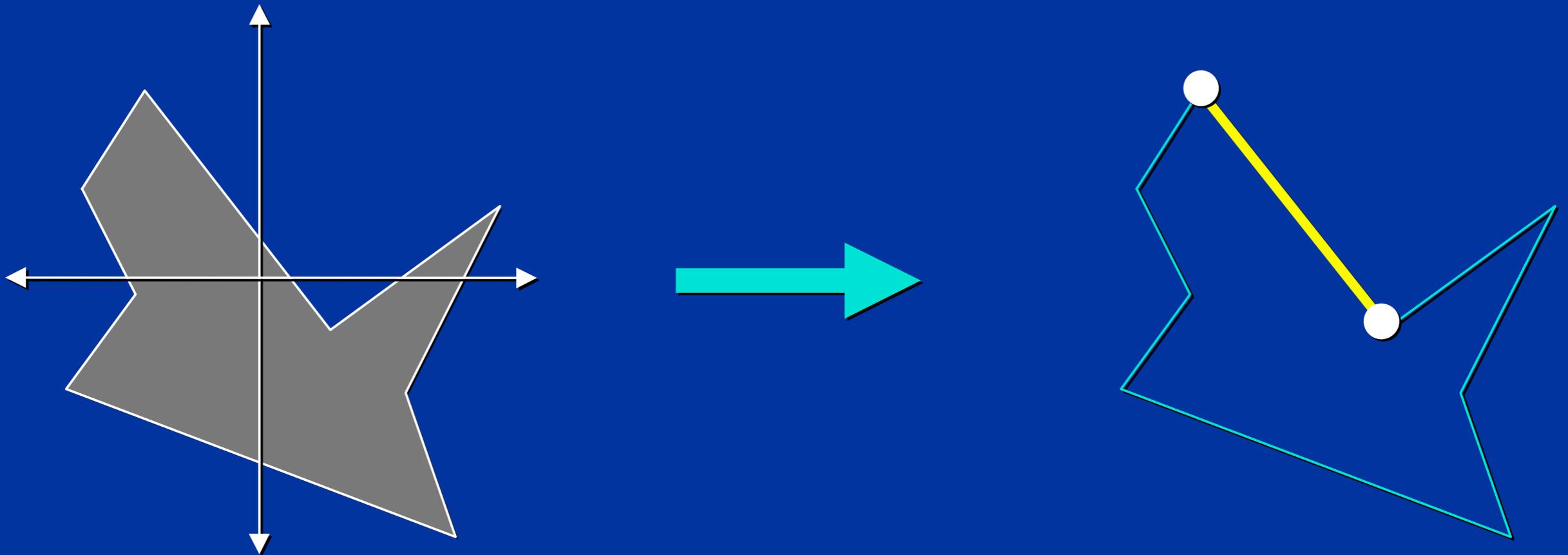
Cons: Bounding box may not be a good fit.  
Inaccurate.

# Approximating $I_{\text{body}}$ : Point Sampling



Pros: Simple, fairly accurate, no B-rep needed.  
Cons: Expensive, requires volume test.

# Computing $I_{\text{body}}$ : Green's Theorem (2x!)



Pros: Simple, exact, no volumes needed.

Cons: Requires boundary representation.

Code: <http://www.acm.org/jgt/papers/Mirtich96>

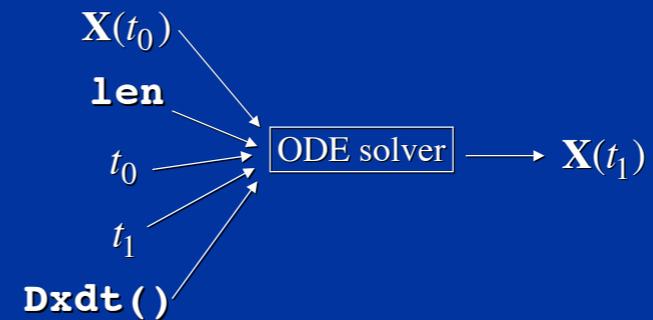
# Summary

## Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{X}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ \mathbf{R}(t) \\ M\mathbf{v}(t) \\ \mathbf{I}(t)\boldsymbol{\omega}(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \boldsymbol{\omega}(t) * \mathbf{R}(t) \\ F(t) \\ \boldsymbol{\tau}(t) \end{pmatrix}$$

$P(t)$  – linear momentum

$L(t)$  – angular momentum



```
void Dxdt(double t, double x[],  
          double xdot[])
```

# What's in the Course Notes

1. Implementation of  $\mathbf{Dxdt}()$  for rigid bodies  
(bookkeeping, data structures, computations)
2. Quaternions—derivations and code
3. Miscellaneous formulas and examples
4. Derivations for force and torque equations,  
center of mass, inertia tensor, rotation  
equations, velocity/acceleration of points

# Example



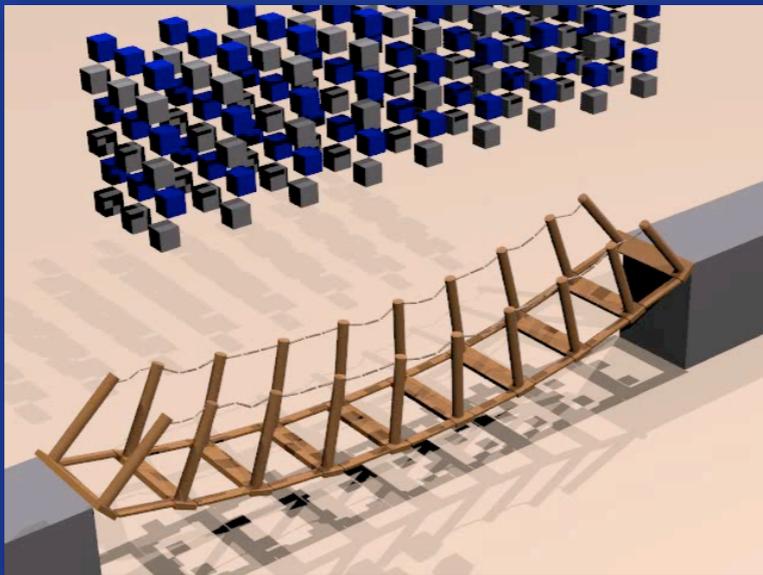
# Example



**These simulations could never have been created by hand.**

# Question

- **What Kind of Collisions Are Possible?**
  - **Geometrically?**
  - **Physically?**



- **How can these be detected?**
- **What algorithm can handle them?**