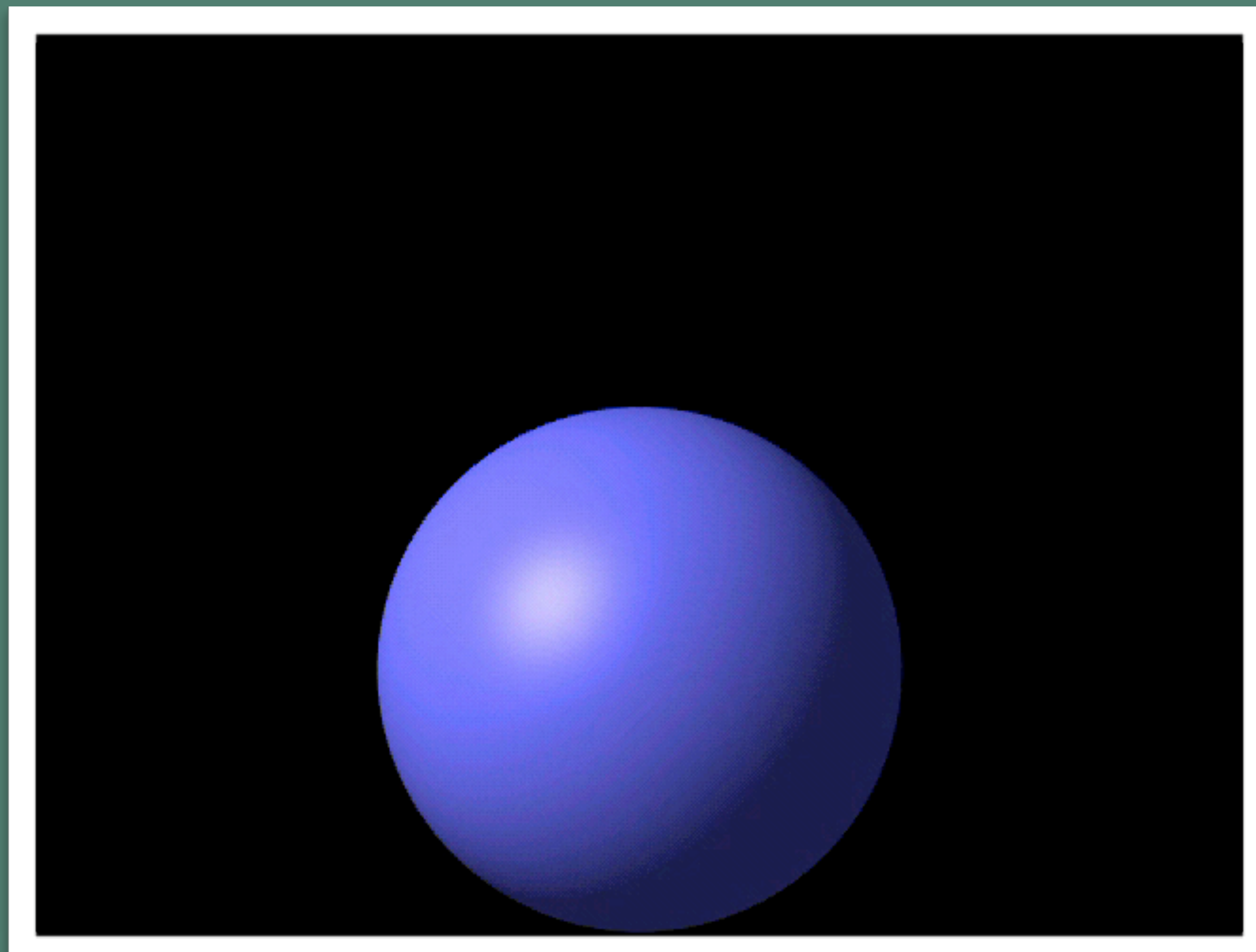
Stable PDE Fluids



Adrien Treuille

Overview

- Last Week's Question: **Advection**
- Fluid Particles (recap)
- From Particles to PDEs
- The Navier Stokes Equations
- Stable Fluids Implementation
- Questions

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Answer to Previous Question

Recall

$$f(x, t) = g(x - t)$$

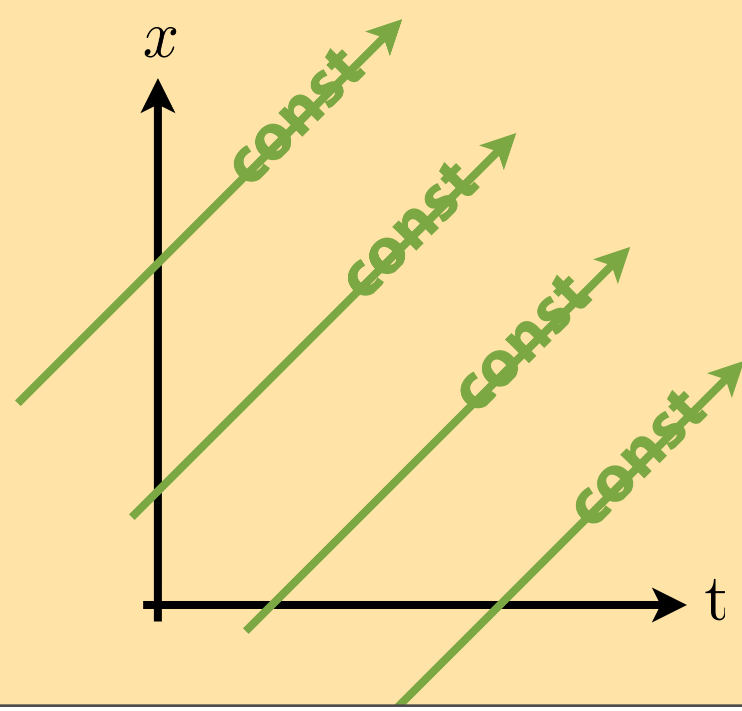
Answer to Previous Question

Recall

$$f(x,t) = g(x-t)$$

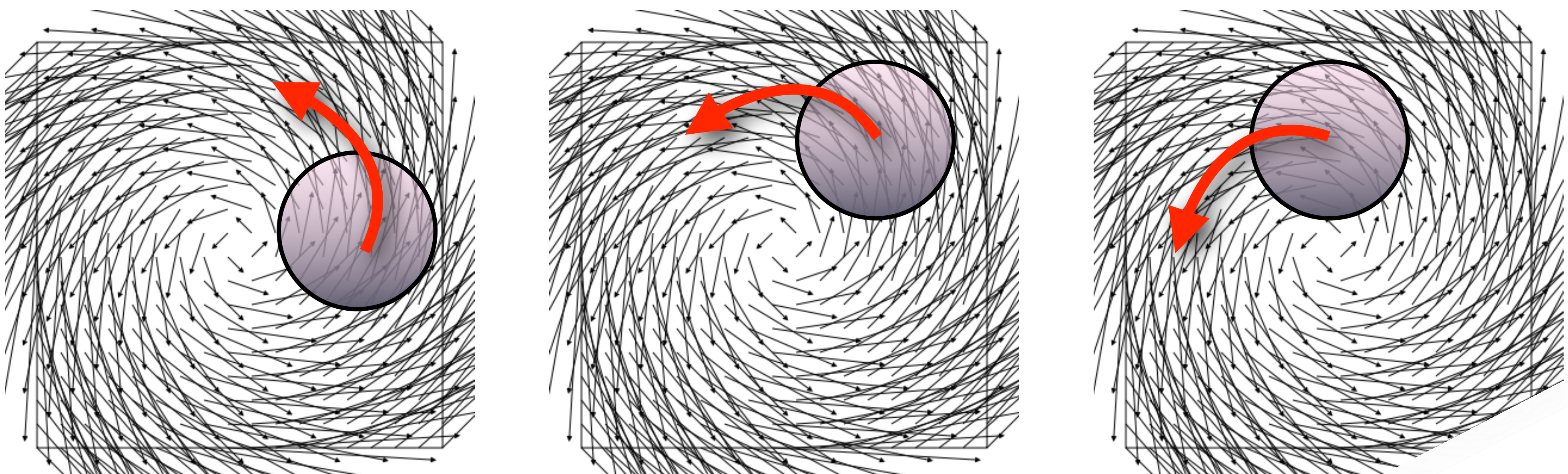
- **Information propagates "to the right"**

$$f(x, t+\Delta t) = f(x-\Delta t, t)$$



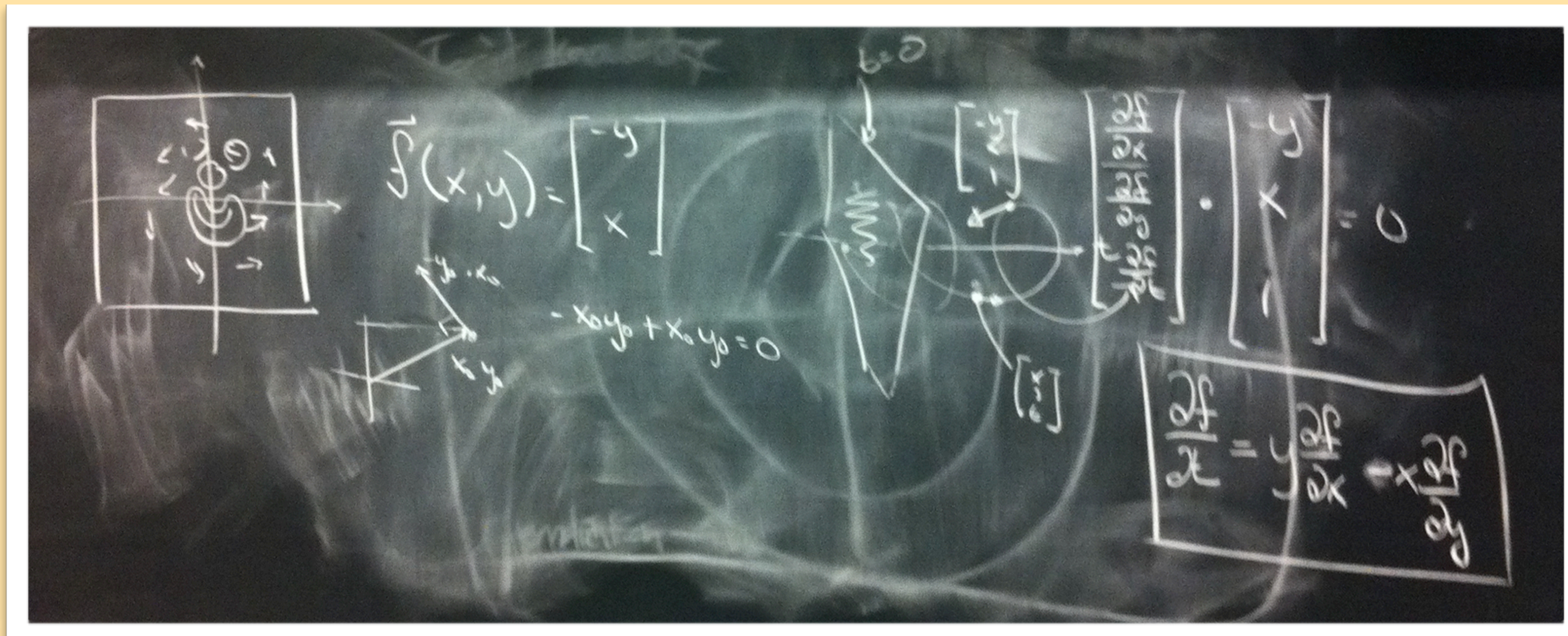
Answer to Previous Question

What about a rotational field?



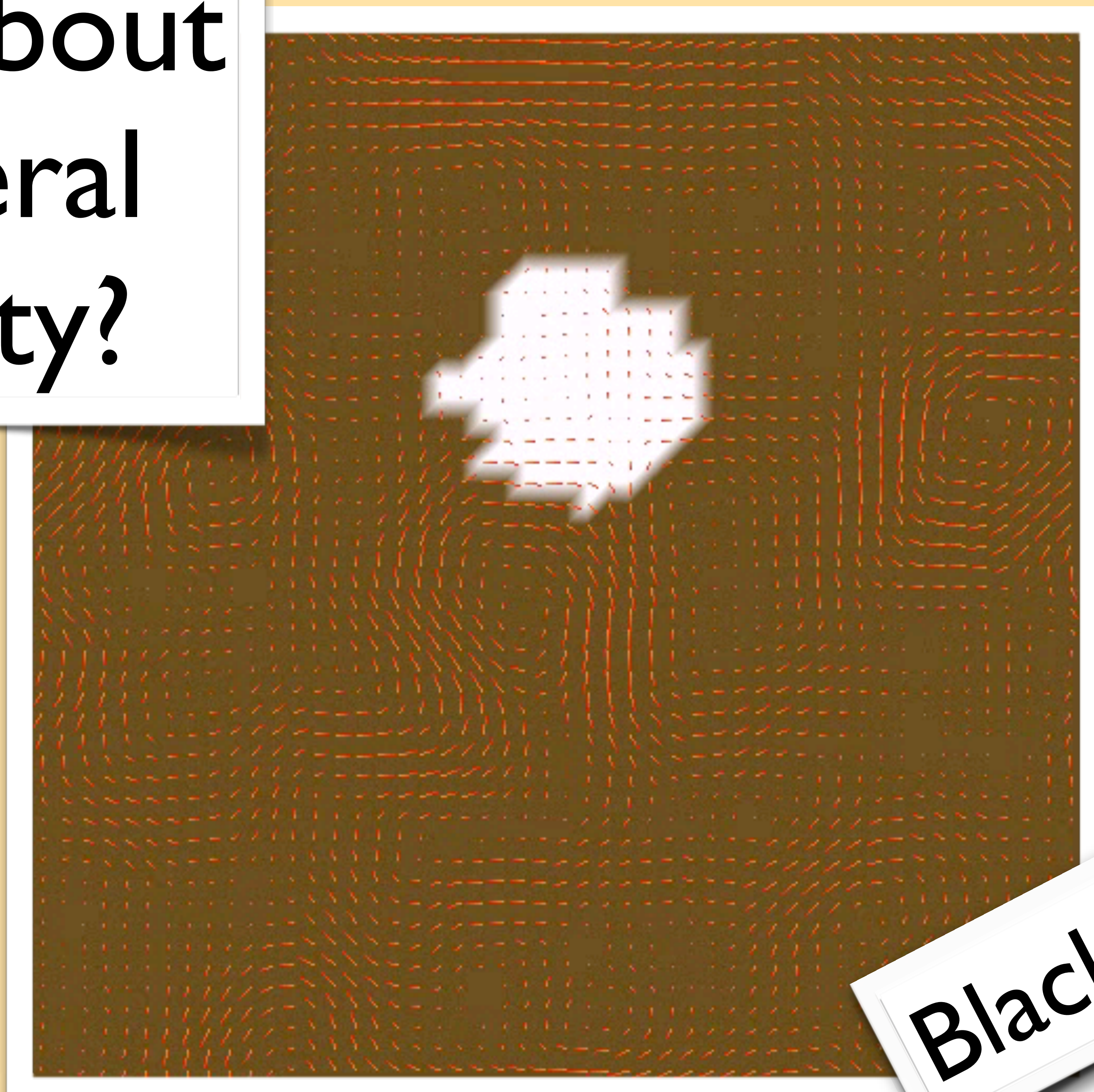
Blackboard

Answer to Previous Question



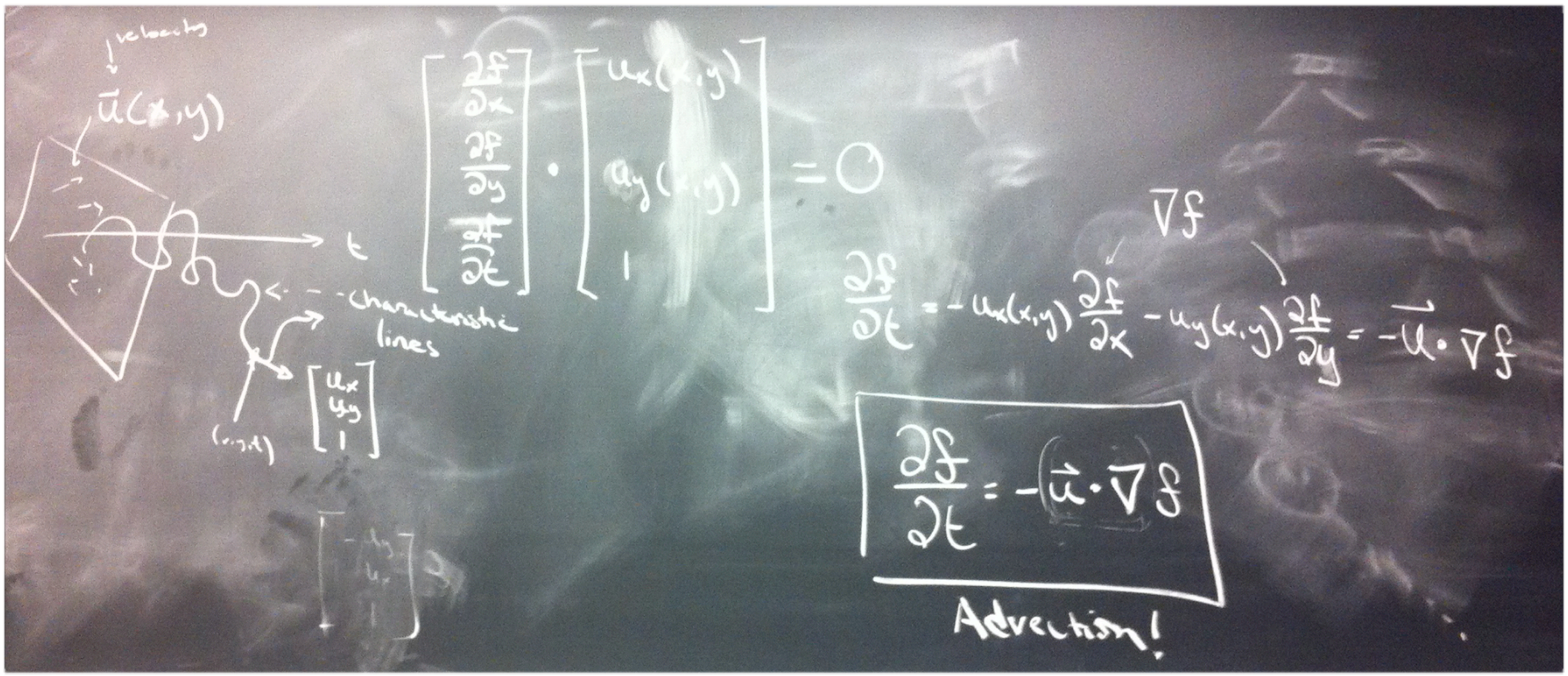
Answer to Previous Question

What about a general velocity?



Blackboard

Answer to Previous Question



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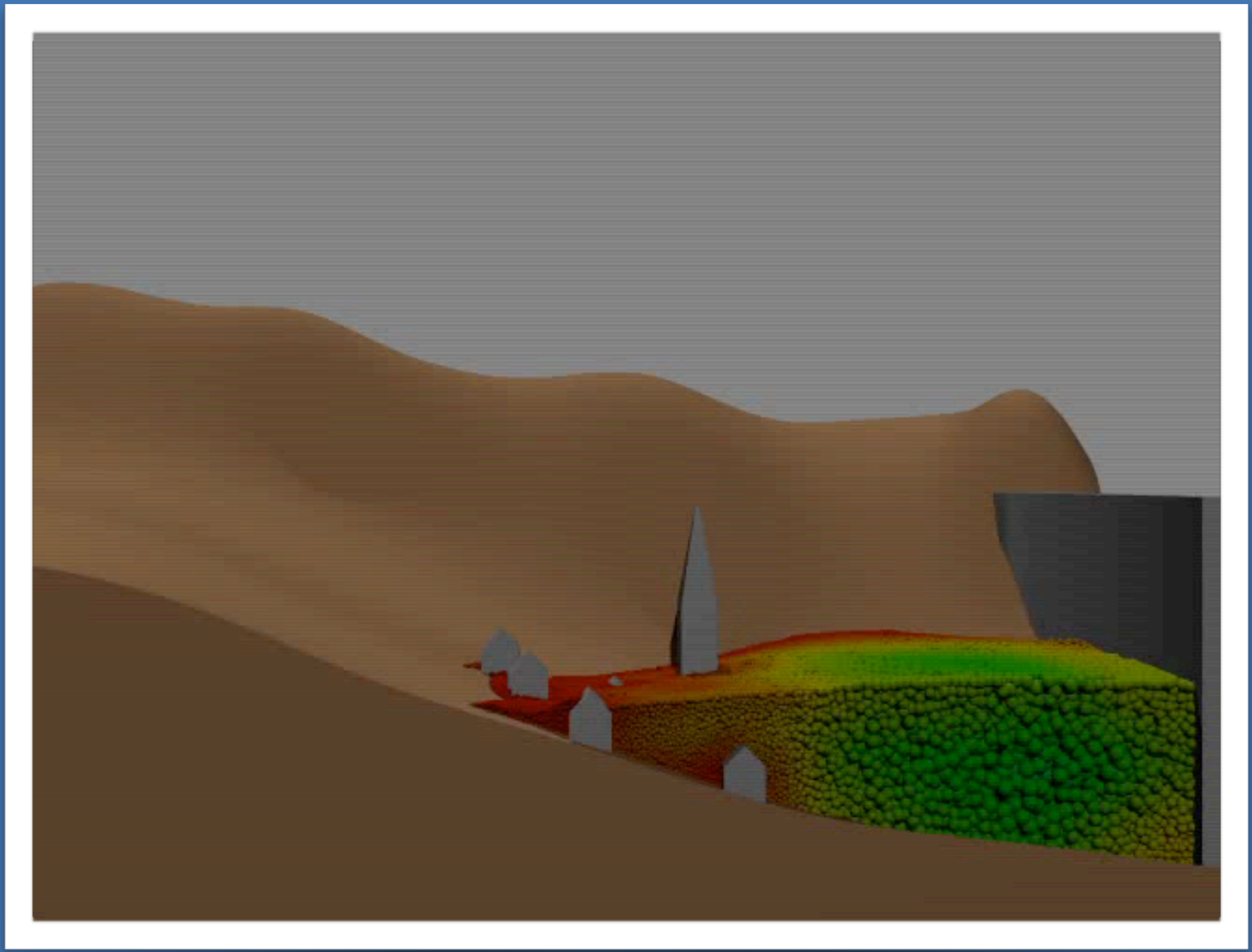
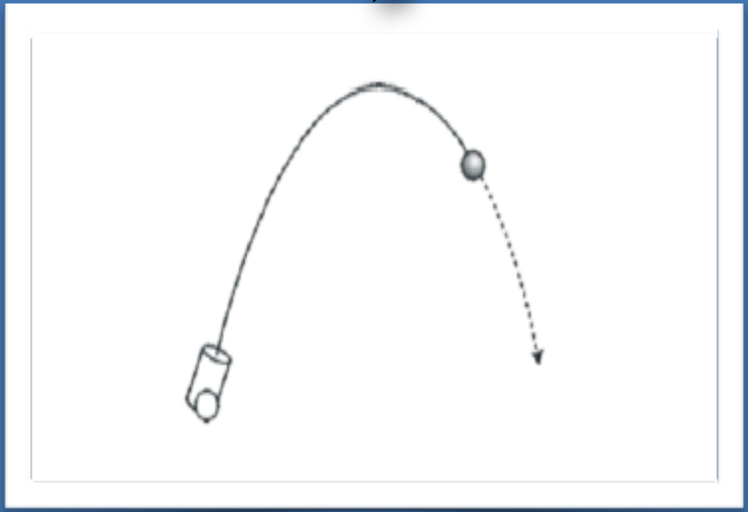
Particle-based Fludis



Fluid Forces

$$\rho \dot{\mathbf{v}} = \mathbf{f}_{\text{gravity}} + \mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{viscosity}}$$



Gravity

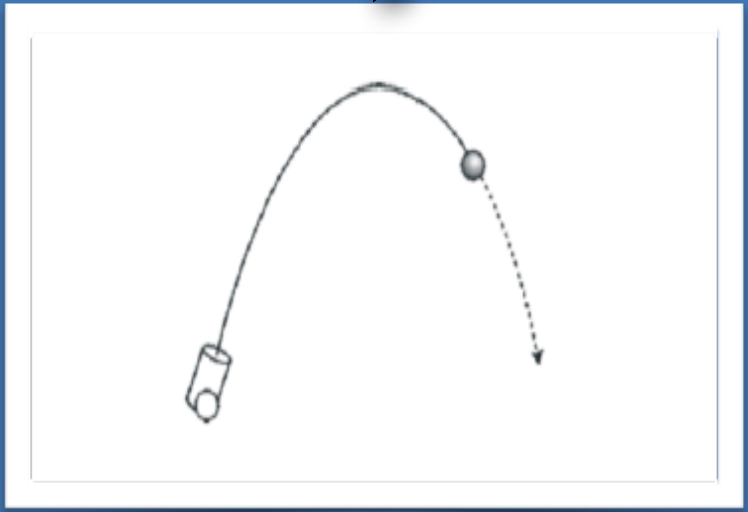
Resistance to Compression

Viscosity

Fluid Forces

$\mathbf{f}$

$$\rho\dot{\mathbf{v}} = \rho\mathbf{g} \quad -\nabla p \quad + \quad \mu\nabla^2\mathbf{v}$$



Gravity

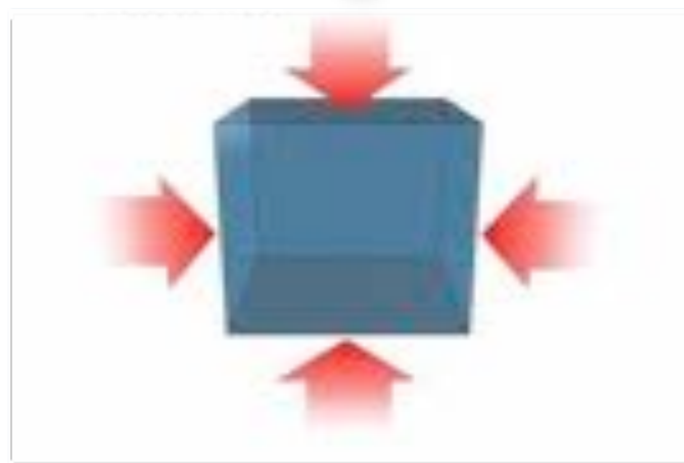
Resistance to Compression

Viscosity

Fluid Forces

$$\rho\dot{\mathbf{v}} = \mathbf{f} \quad -\nabla p \quad + \quad \mu\nabla^2\mathbf{v}$$

External Forces



Resistance to Compression

Viscosity

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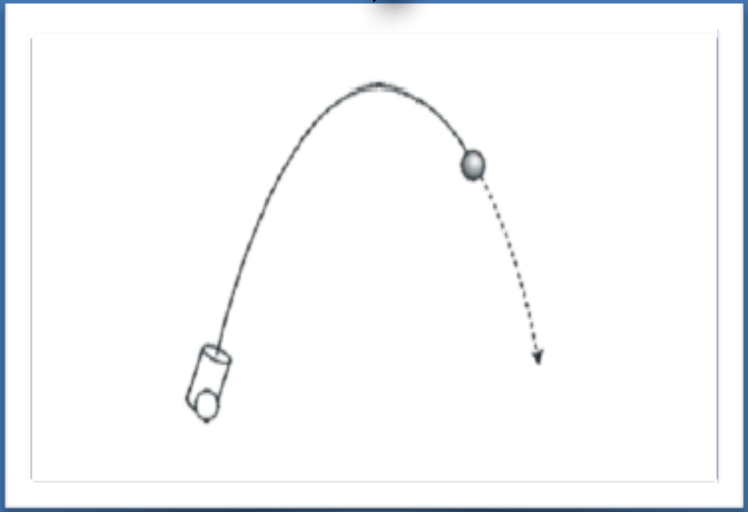
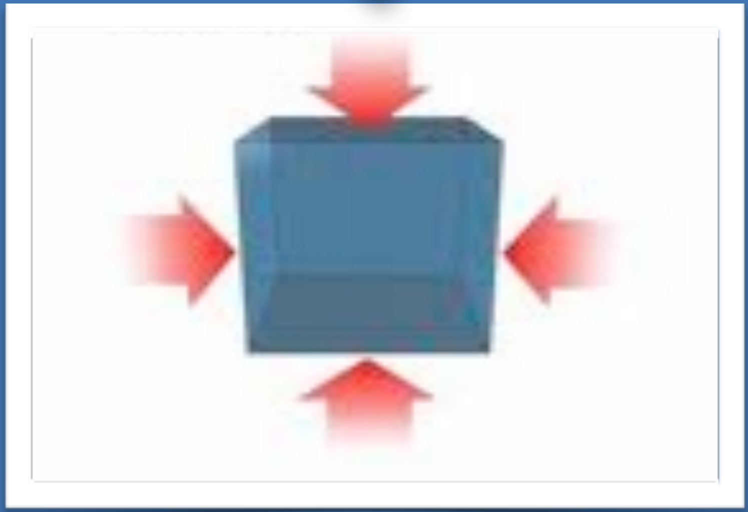
From Particles to PDES

$$\rho\dot{\mathbf{v}} = \mathbf{f} - \nabla p + \mu\nabla^2\mathbf{v}$$



External Forces



Resistance to Compression

Viscosity

From Particles to PDES

$\rho\dot{\mathbf{v}}$

From Particles to PDES

$$\rho \dot{\mathbf{u}}$$

From Particles to PDES

$$\rho\dot{\mathbf{u}}$$

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$$

From Particles to PDES

$$\rho\dot{\mathbf{u}}$$

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \rho\left(\frac{\partial\mathbf{u}}{\mathrm{d}\mathbf{x}}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial\mathbf{u}}{\partial t}\right)$$

From Particles to PDES

$$\rho\dot{\mathbf{u}}$$

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \rho\left(\frac{\partial\mathbf{u}}{\mathrm{d}\mathbf{x}}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial\mathbf{u}}{\partial t}\right) = \rho\left(\frac{\partial\mathbf{u}}{\mathrm{d}\mathbf{x}}\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t}\right)$$

From Particles to PDES

$$\rho \dot{\mathbf{u}}$$

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \rho \left(\frac{\partial \mathbf{u}}{\mathrm{d}\mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial \mathbf{u}}{\partial t} \right) = \rho \left(\frac{\partial \mathbf{u}}{\mathrm{d}\mathbf{x}} \mathbf{u} + \frac{\partial \mathbf{u}}{\partial t} \right)$$

$$= \rho \left(\mathbf{u} \cdot \nabla \mathbf{u} + \frac{\partial \mathbf{u}}{\partial t} \right)$$

From Particles to PDES

$$\rho\dot{\mathbf{u}}$$

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \rho\left(\frac{\partial\mathbf{u}}{\mathrm{d}\mathbf{x}}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial\mathbf{u}}{\partial t}\right) = \rho\left(\frac{\partial\mathbf{u}}{\mathrm{d}\mathbf{x}}\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t}\right)$$

$$= \rho\left(\mathbf{u}\cdot\nabla\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t}\right)$$

From Particles to PDES

$$\rho \left(\mathbf{u} \cdot \nabla \mathbf{u} + \frac{\partial \mathbf{u}}{\partial t} \right)$$

From Particles to PDES

$$\rho\left(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{\partial\mathbf{u}}{\partial t}\right)=-\nabla p+\mu\nabla^2\mathbf{u}+f$$

From Particles to PDES

$$\rho\left(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{\partial\mathbf{u}}{\partial t}\right)=-\nabla p+\mu\nabla^2\mathbf{u}+f$$

$$\frac{\partial\mathbf{u}}{\partial t}=\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p+\mu\nabla^2\mathbf{u}+f$$

From Particles to PDES

$$\rho\left(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{\partial\mathbf{u}}{\partial t}\right)=-\nabla p+\mu\nabla^2\mathbf{u}+f$$

$$\frac{\partial\mathbf{u}}{\partial t}=\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p+\mu\nabla^2\mathbf{u}+f$$

$$\text{s.t. }\nabla\cdot\mathbf{u}=0$$

From Particles to PDES

$$\rho\left(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{\partial\mathbf{u}}{\partial t}\right)=-\nabla p+\mu\nabla^2\mathbf{u}+f$$

$$\frac{\partial\mathbf{u}}{\partial t}=\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p+\mu\nabla^2\mathbf{u}+f$$

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Incompressible Navier-Stokes Equations!

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Navier Stokes Equations

Navier Stokes Equations

- Density

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

Navier Stokes Equations

- Density

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

Navier Stokes Equations

- Density

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

Navier Stokes Equations

- Density

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

- Velocity

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

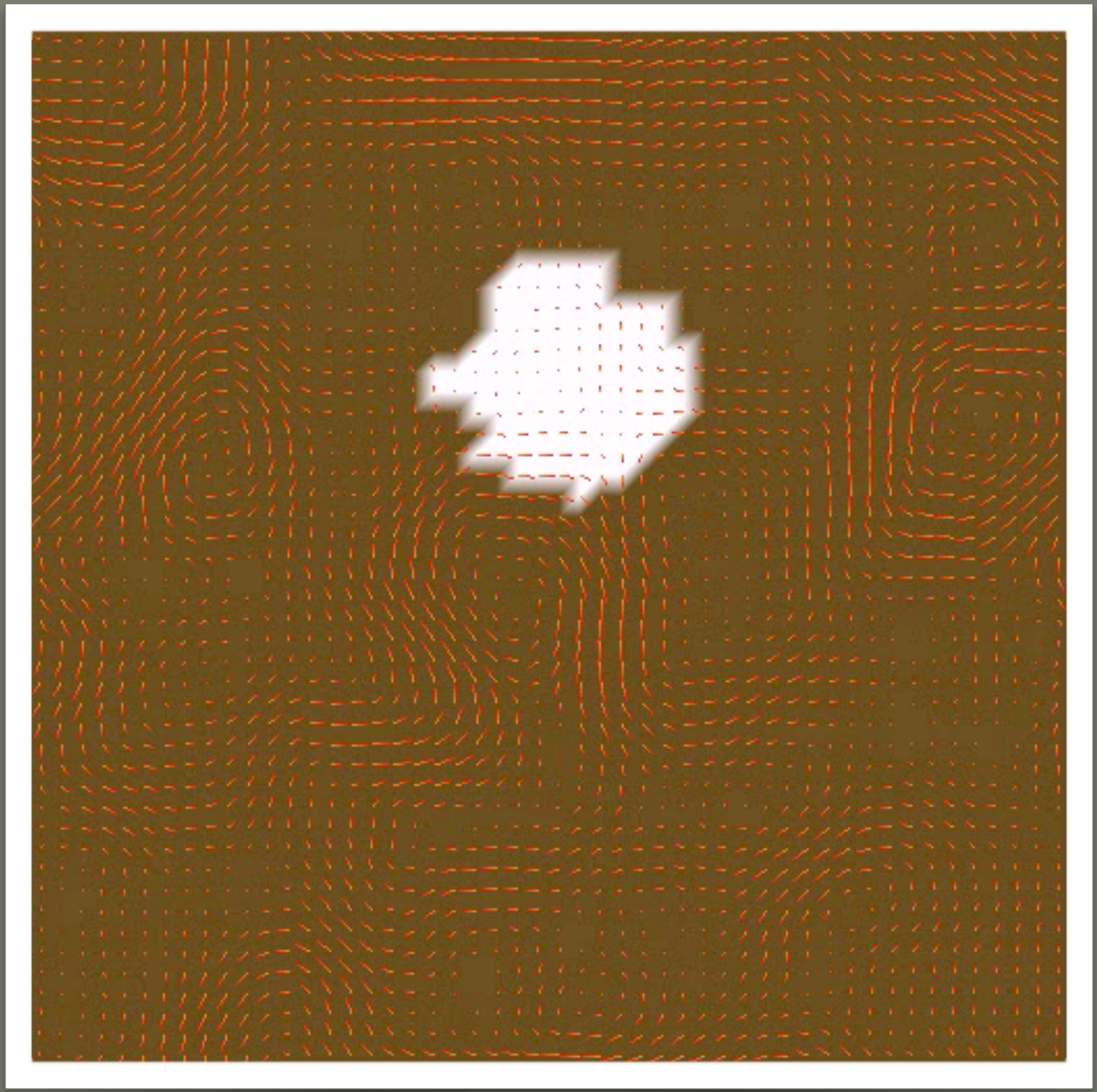
s.t. $\nabla \cdot \mathbf{u} = 0$

Density Advection

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

Video: Density Advection

Density Advection



Velocity Advection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

Video: Velocity Advection

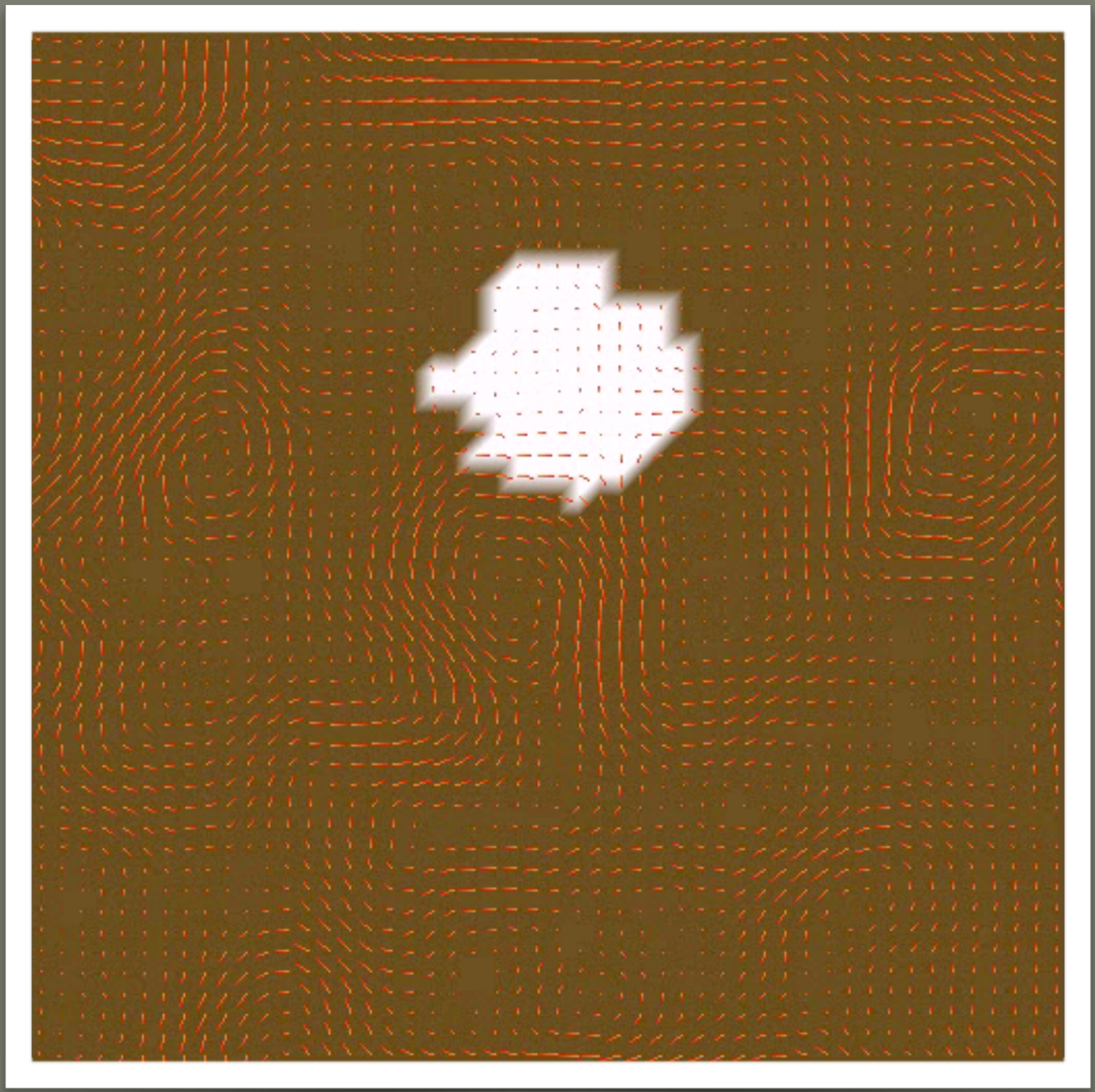
Velocity Advection

$$\frac{\partial \mathbf{u}}{\partial t} = \boxed{-(\mathbf{u} \cdot \nabla)}\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

Video: Velocity Advection

Density and Velocity Advection



Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

s.t. $\boxed{\nabla \cdot \mathbf{u} = 0}$

Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$\boxed{\text{s.t. } \nabla \cdot \mathbf{u} = 0}$$

(divergence)

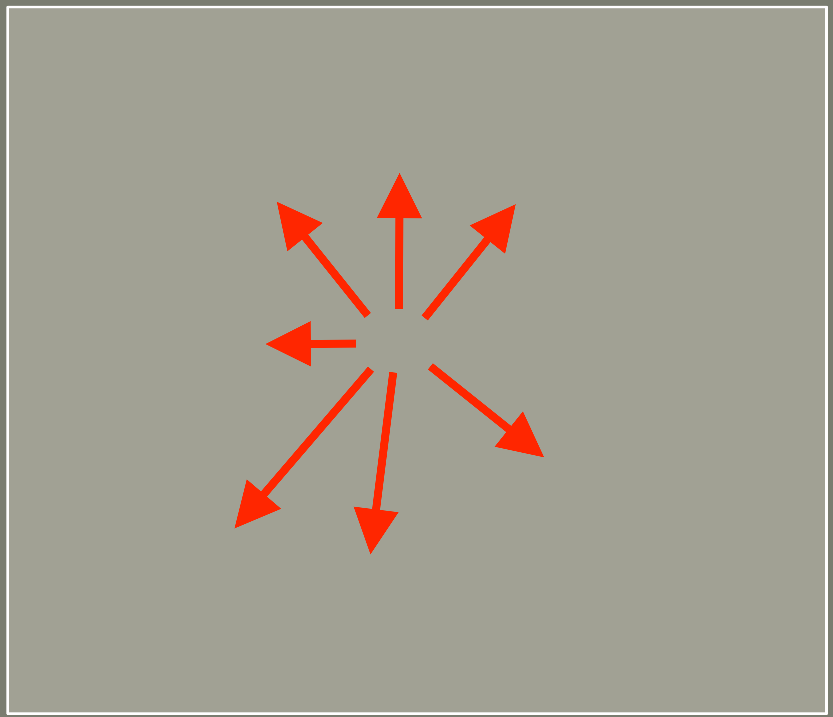
Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$\boxed{\text{s.t. } \nabla \cdot \mathbf{u} = 0}$

(divergence)

Div > 0



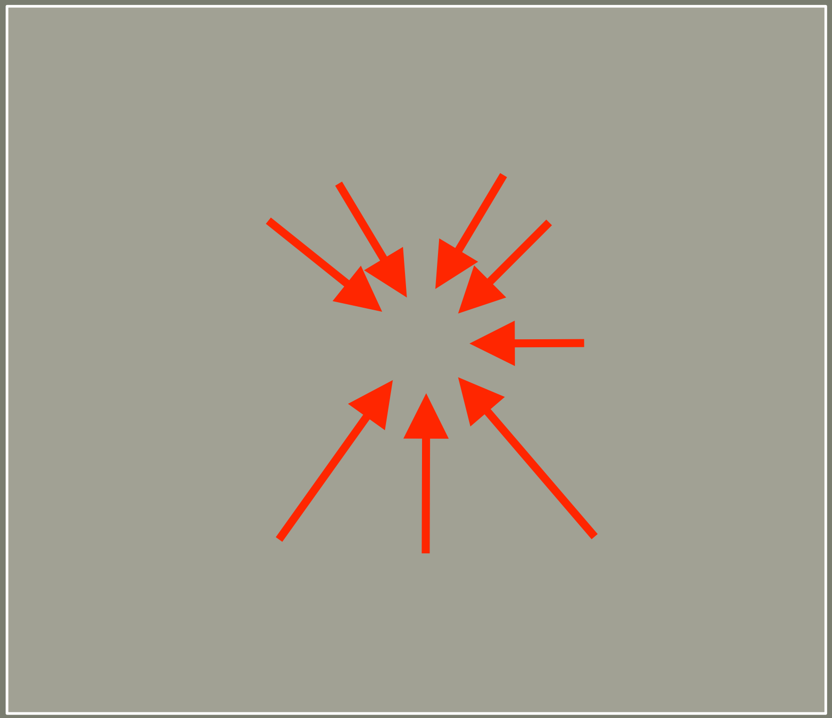
Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$\boxed{\text{s.t. } \nabla \cdot \mathbf{u} = 0}$$

(divergence)

Div < 0



Projection

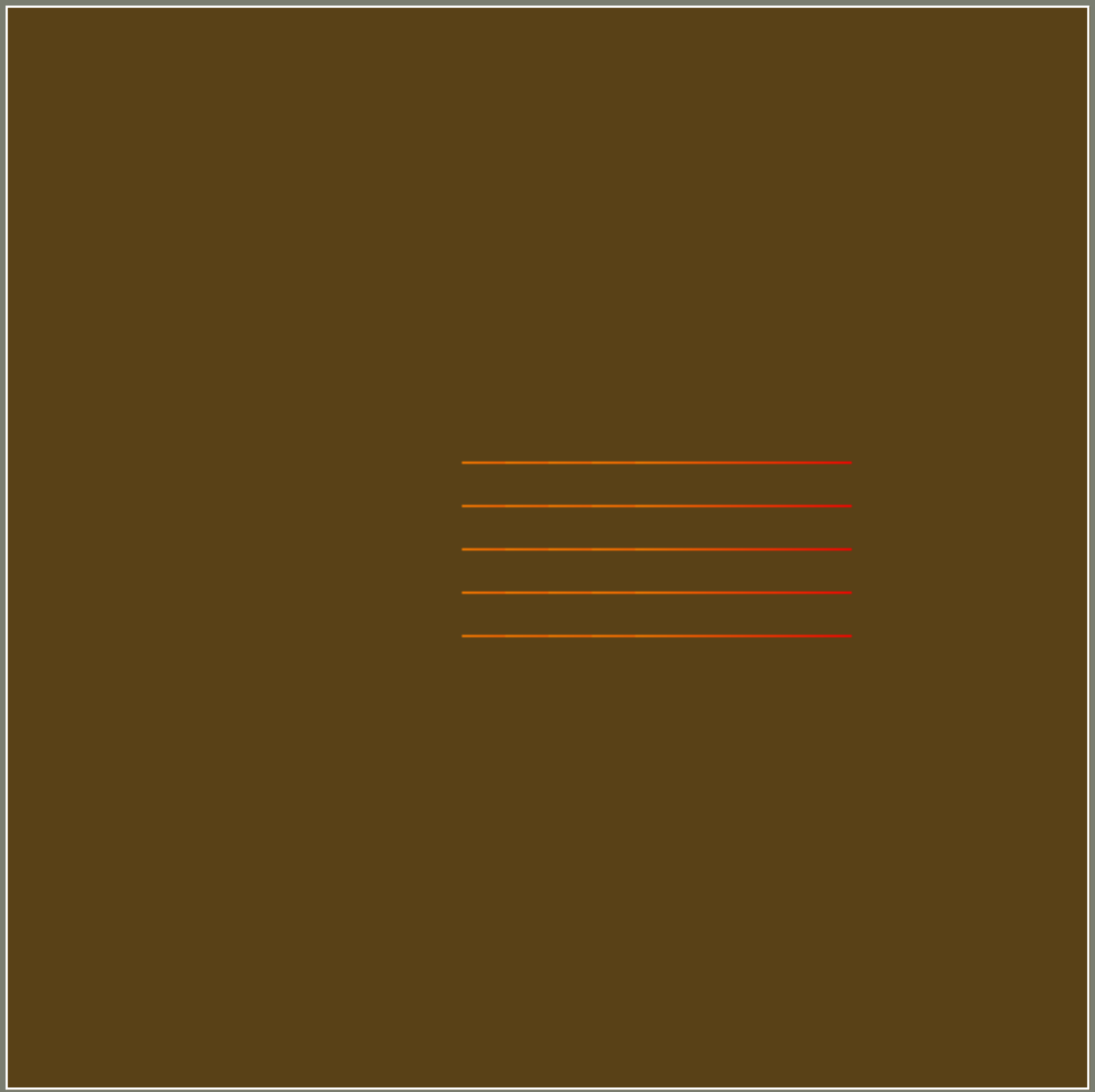
$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$\boxed{\text{s.t. } \nabla \cdot \mathbf{u} = 0}$$

(divergence)

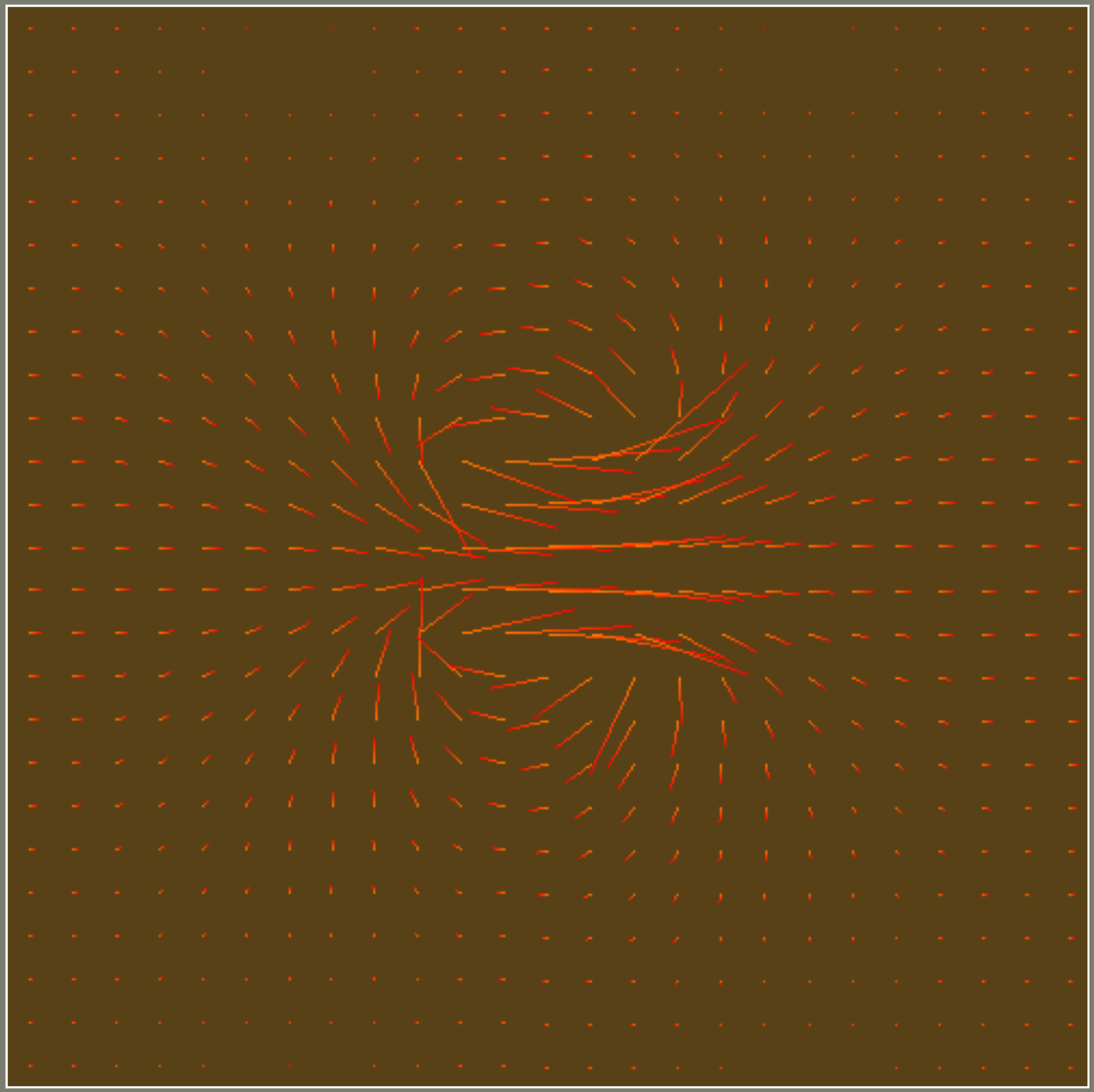
Div = 0 ?

Projection



$\text{Div} \neq 0$

Projection



Div = 0

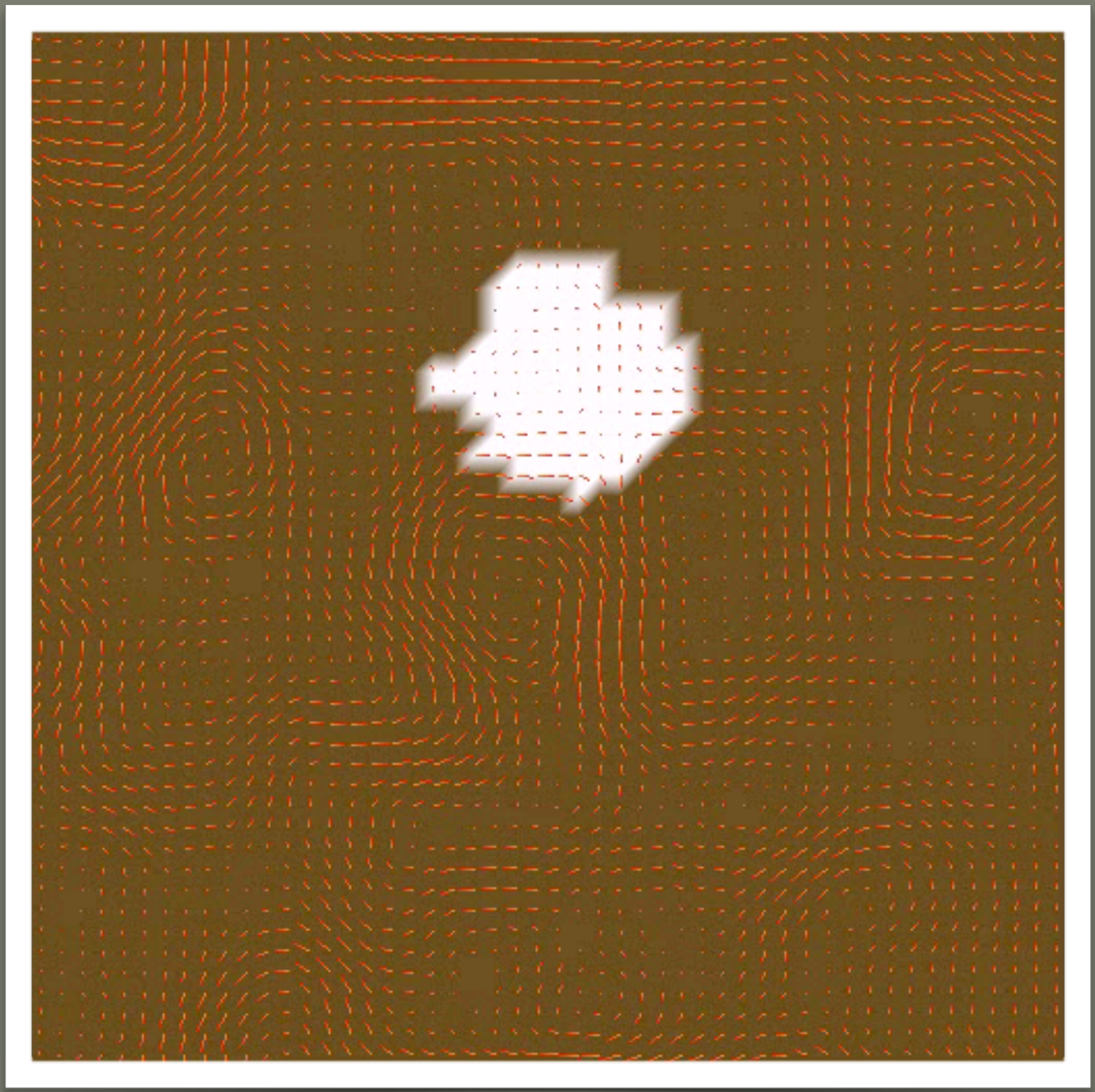
Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$\boxed{\text{s.t. } \nabla \cdot \mathbf{u} = 0}$$

Video: Velocity Advection and Projection

Advection + Projection



Diffusion

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + \boxed{s\nabla^2\mathbf{u}} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

External Forces

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \boxed{\mathbf{f}}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

External Forces

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \boxed{\mathbf{f}}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

- Gravity

External Forces

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \boxed{\mathbf{f}}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

- Gravity
- Heat

External Forces

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s.t. $\nabla \cdot \mathbf{u} = 0$

- Gravity
- Heat
- Surface Tension

External Forces

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \boxed{\mathbf{f}}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

- Gravity
- Heat
- Surface Tension
- User-Created Forces (stirring coffee)

Physics Recap

Physics Recap

- Physical quantities represented as fields.

Physics Recap

- Physical quantities represented as fields.
- PDE describes the dynamics.
  - –explains what we see in here…

- Much $$$ for analytic solution!

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Simulation Representation

Simulation Representation

- Recall we're dealing with *fields*:

$$\rho : \Omega \to [0, 1]$$
(density)

$$\mathbf{u} : \Omega \to \mathbf{R}^3$$
(velocity)

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- Grid Representation

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Simulation Representation

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(density) (velocity)

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- Grid Representation

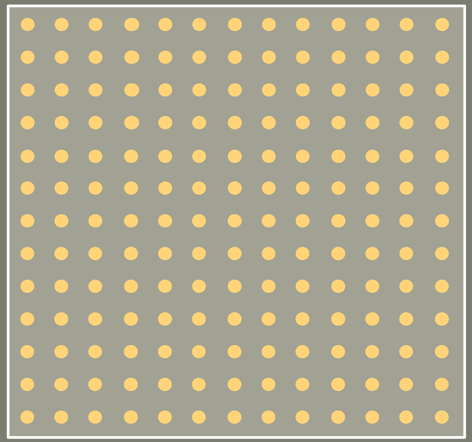
Simulation Representation

- Recall we're dealing with *fields*:

$$\rho : \Omega \to [0, 1] \qquad\qquad \mathbf{u} : \Omega \to \mathbf{R}^3$$

(density) (velocity)

- Grid Representation



  - Each grid cell represents integral over underlying quantities
  - Derivatives Easy to Implement

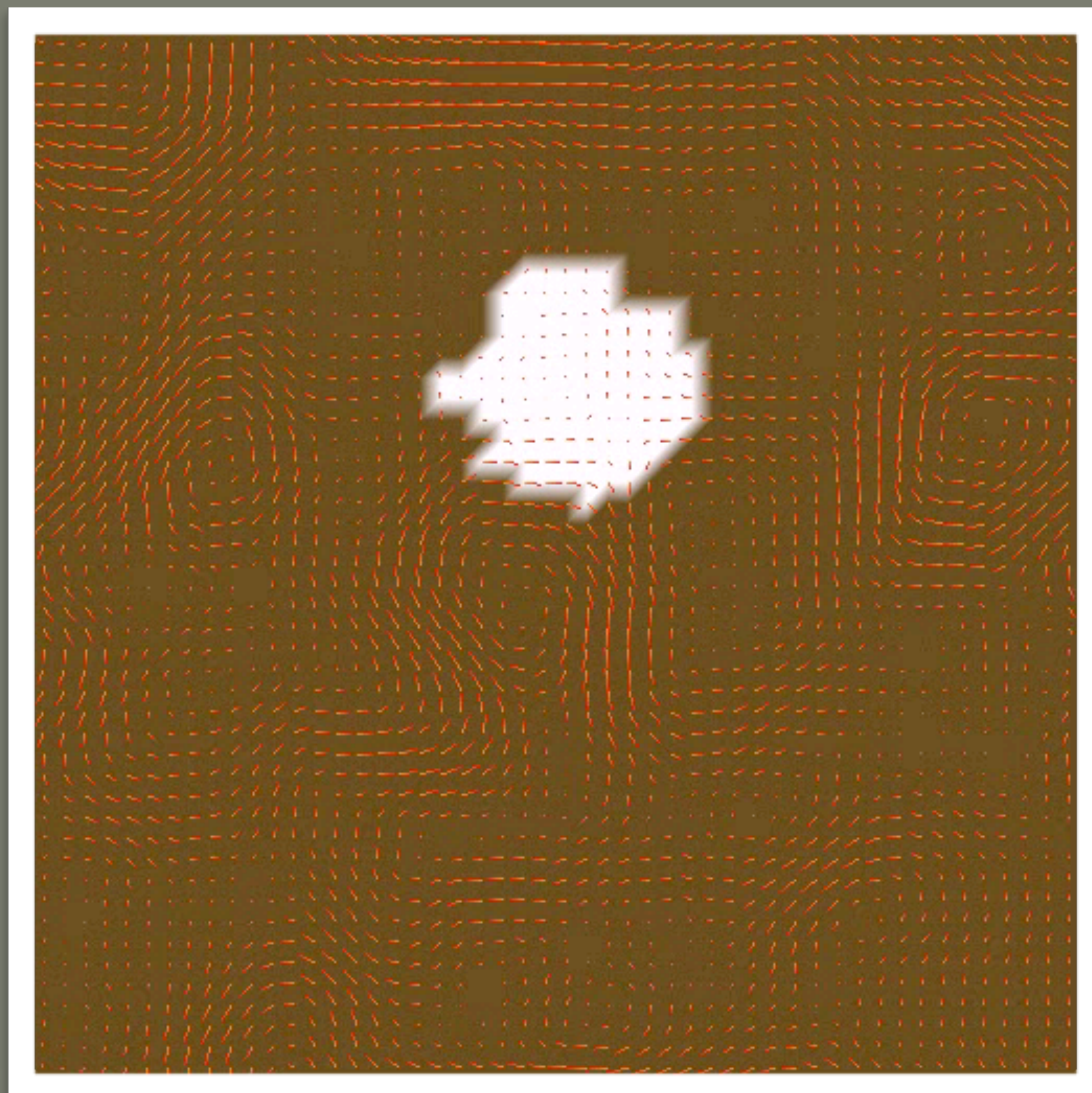
Explicit Integration

- Very simple method to “implement” physics

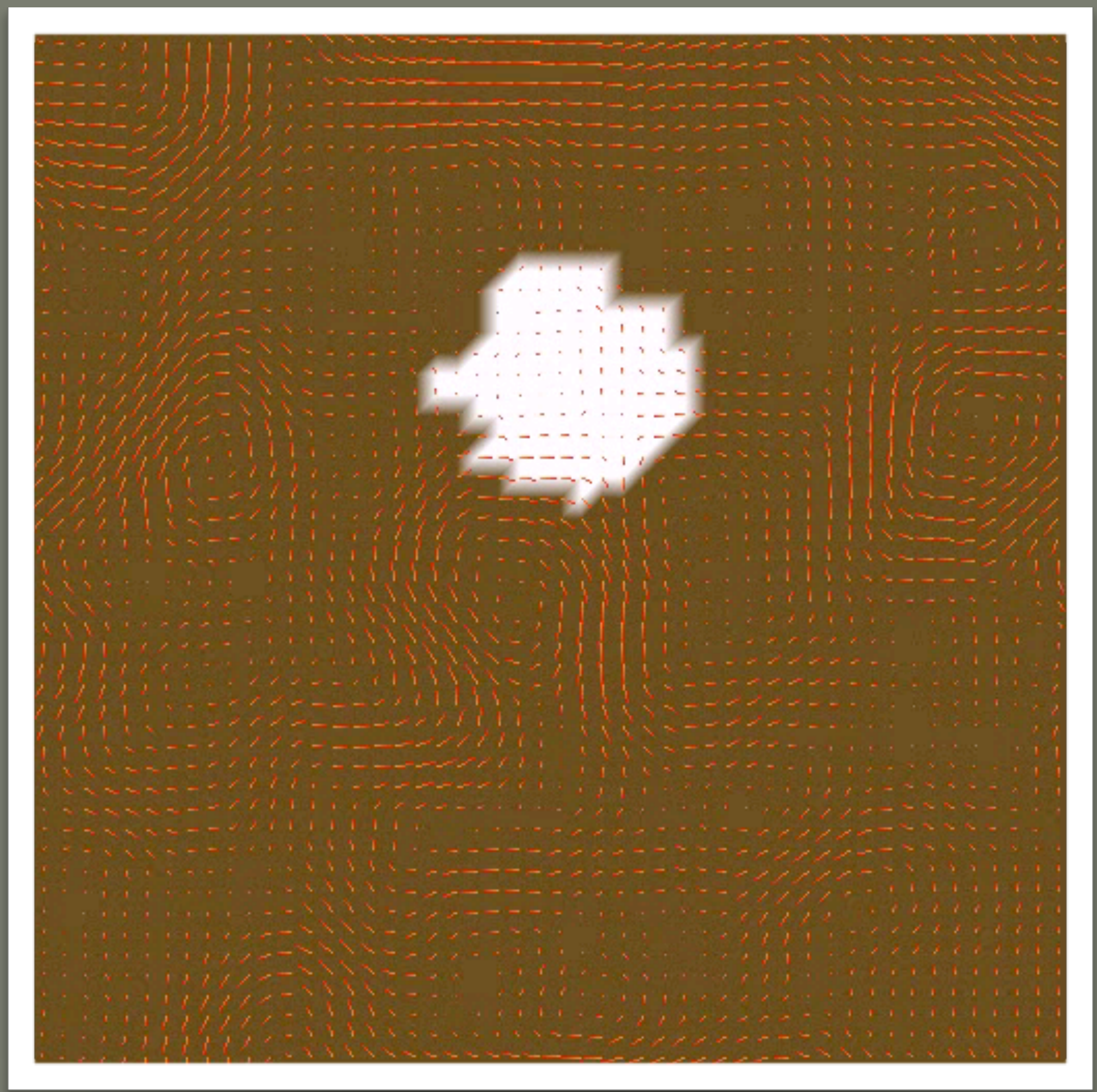
$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$x(t + \Delta t) \approx x(t) + (\Delta t) f(x(t))$$

Explicit Integration



Stable Fluids



Splitting Methods

Splitting Methods

- Suppose we had a system:

$$\frac{\partial x}{\partial t} = f(x) = g(t) + h(t)$$

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Splitting Methods

- Suppose we had a system:

$$\frac{\partial x}{\partial t} = f(x) = g(t) + h(t)$$

- …and we define a *simulation* S_f.

$$S_f(x, \Delta t) \ \ : \ \ x(t) \mapsto x(t + \Delta t)$$

Splitting Methods

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$$\frac{\partial x}{\partial t} = f(x) = g(t) + h(t)$$

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- Suppose we had a system:

$$\frac{\partial x}{\partial t} = f(x) = g(t) + h(t)$$

- …and we define a *simulation* S_f.

$$S_f(x, \Delta t) \ \ : \ \ x(t) \mapsto x(t + \Delta t)$$

- Then we *could* define:

$$S_f(x, \Delta t) = S_g(x, \Delta t) \circ S_h(x, \Delta t)$$

Splitting Methods

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

Splitting Methods

$$\frac{\partial \mathbf{u}}{\partial t} = -\boxed{(\mathbf{u} \cdot \nabla)\mathbf{u}} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

Advect

Splitting Methods

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$\boxed{\text{Advect}} \rightarrow \boxed{\text{Project}}$

Splitting Methods

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + \boxed{s\nabla^2 \mathbf{u}} + \mathbf{f}$$

Advect → **Project** → **Diffuse**

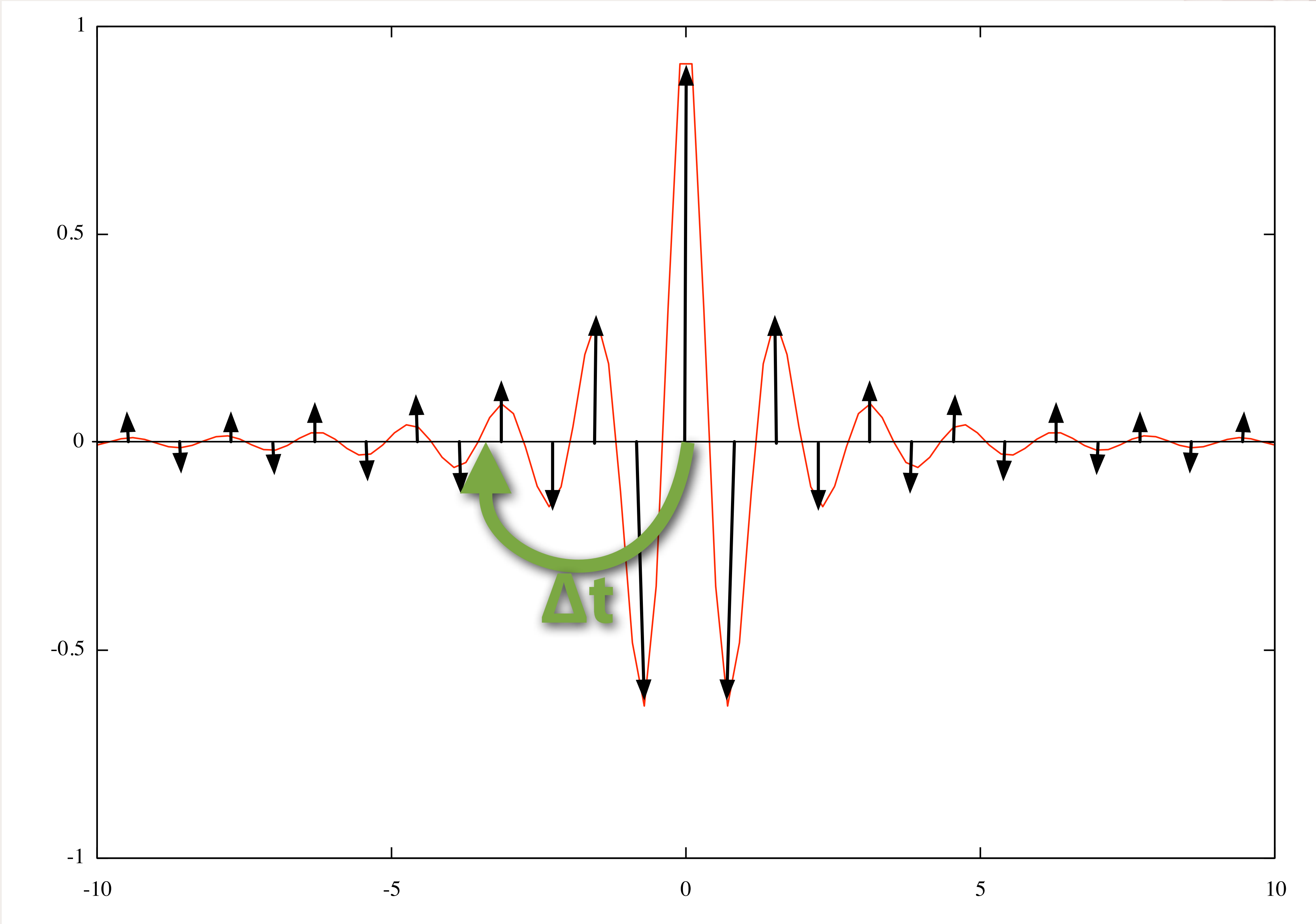
Splitting Methods

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \boxed{\mathbf{f}}$$

Advect → Project → Diffuse → Add Forces

Semi-Lagrangian

$$f(x, t + \Delta t) = f(x - \Delta t, t)$$

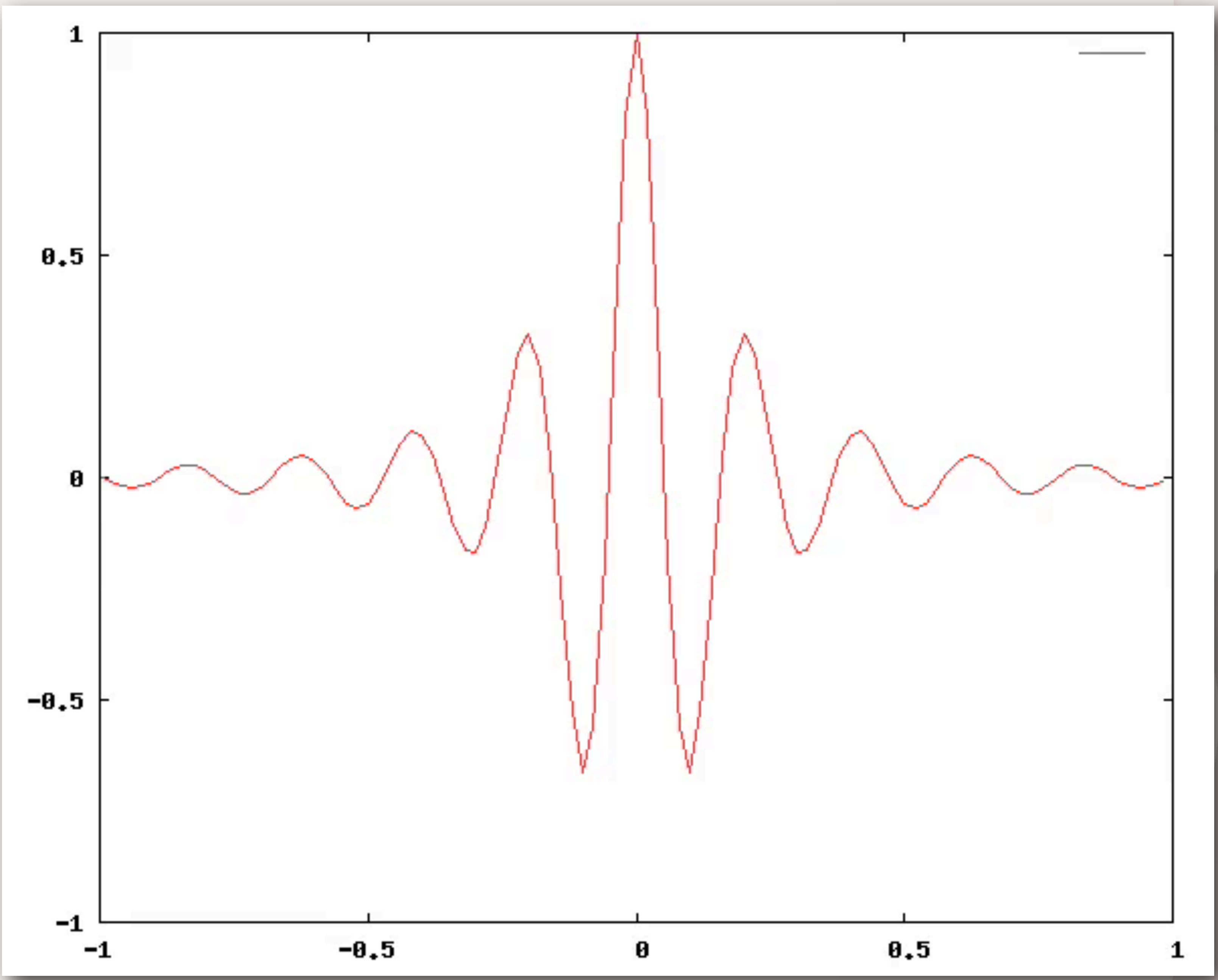


SL Advection

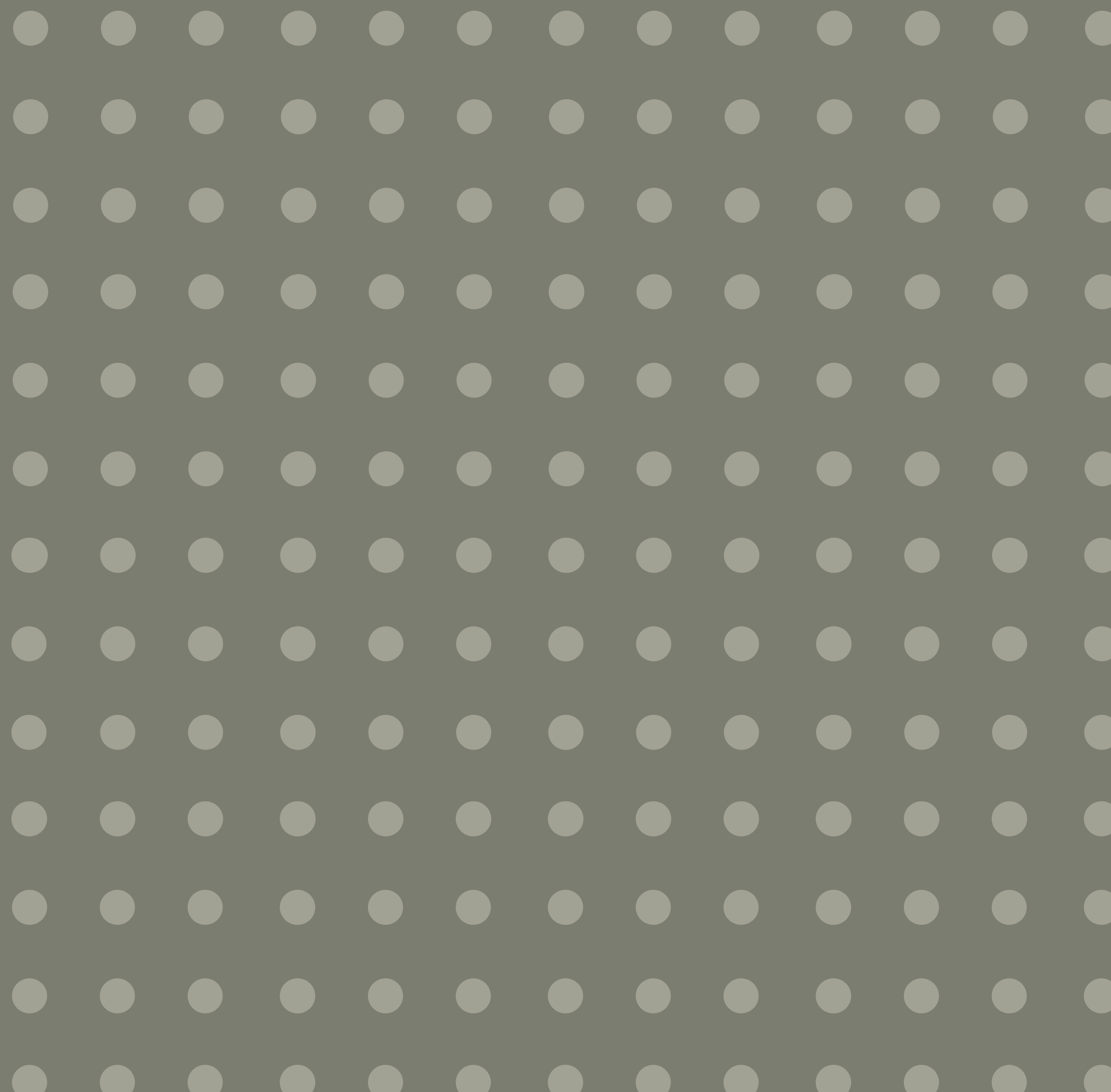
$$f(x, t + \Delta t) = f(x - \Delta t, t)$$

SL Advection

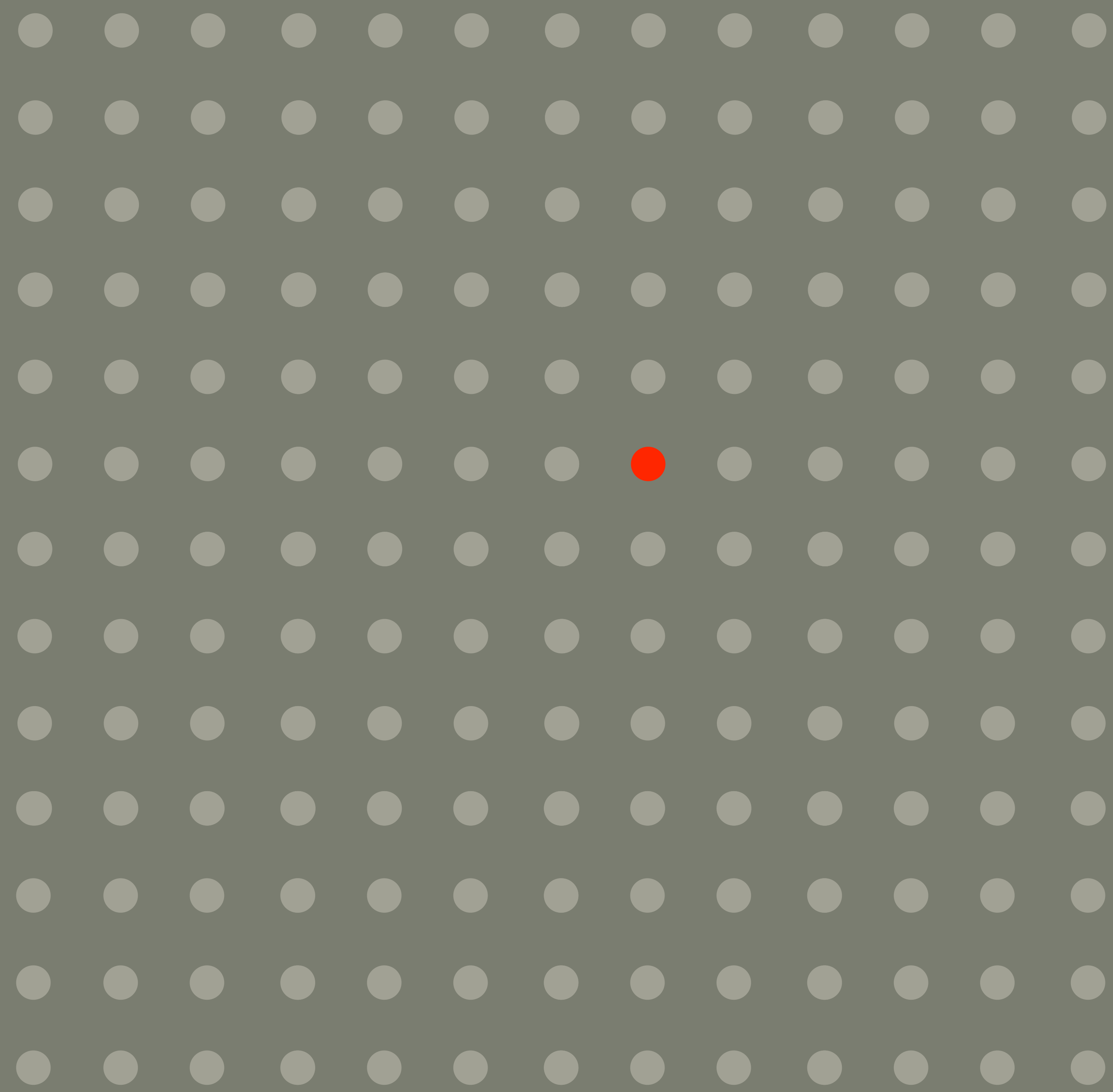
$$f(x, t + \Delta t) = f(x - \Delta t, t)$$



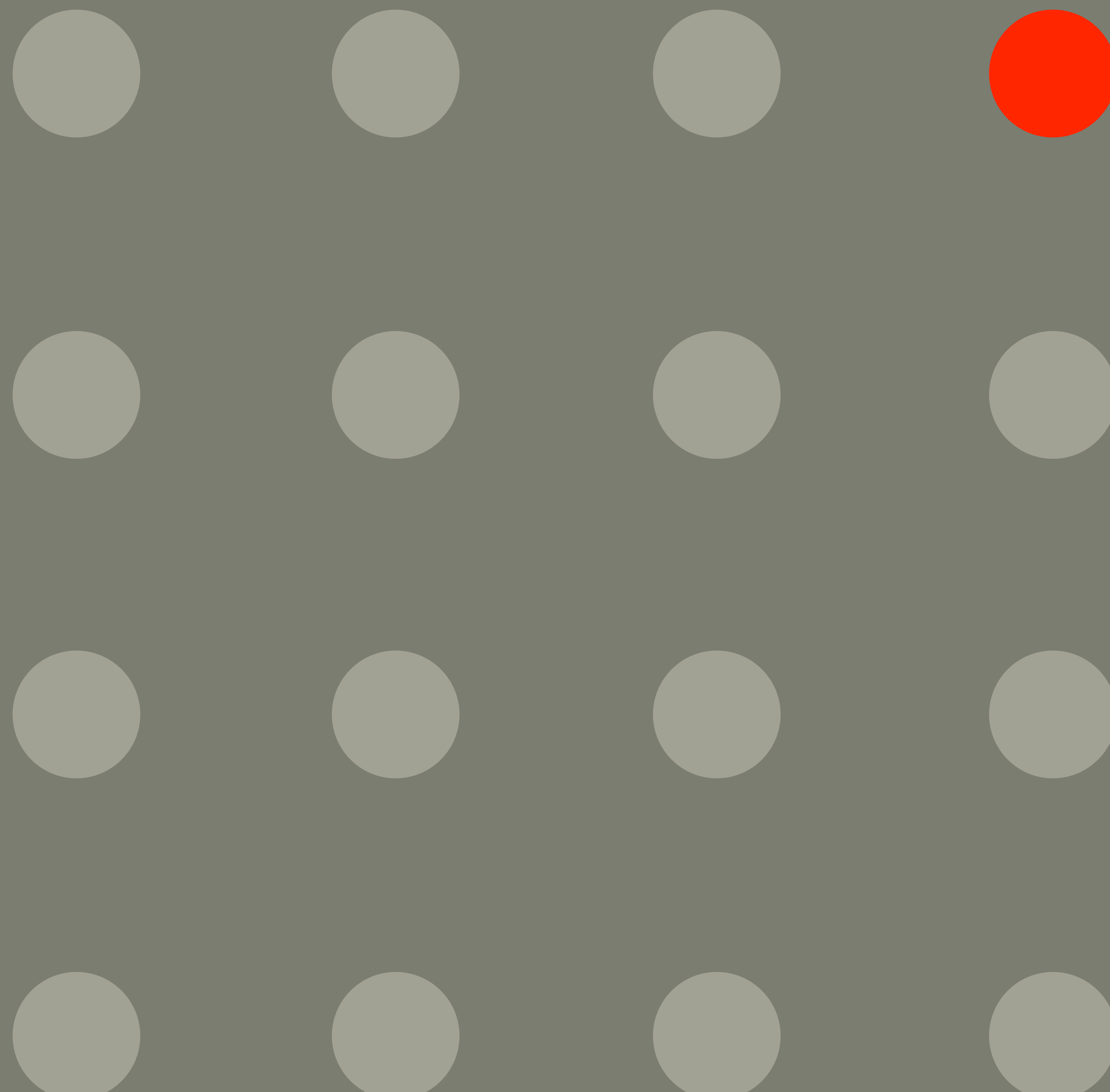
Advection



Advection



Advection



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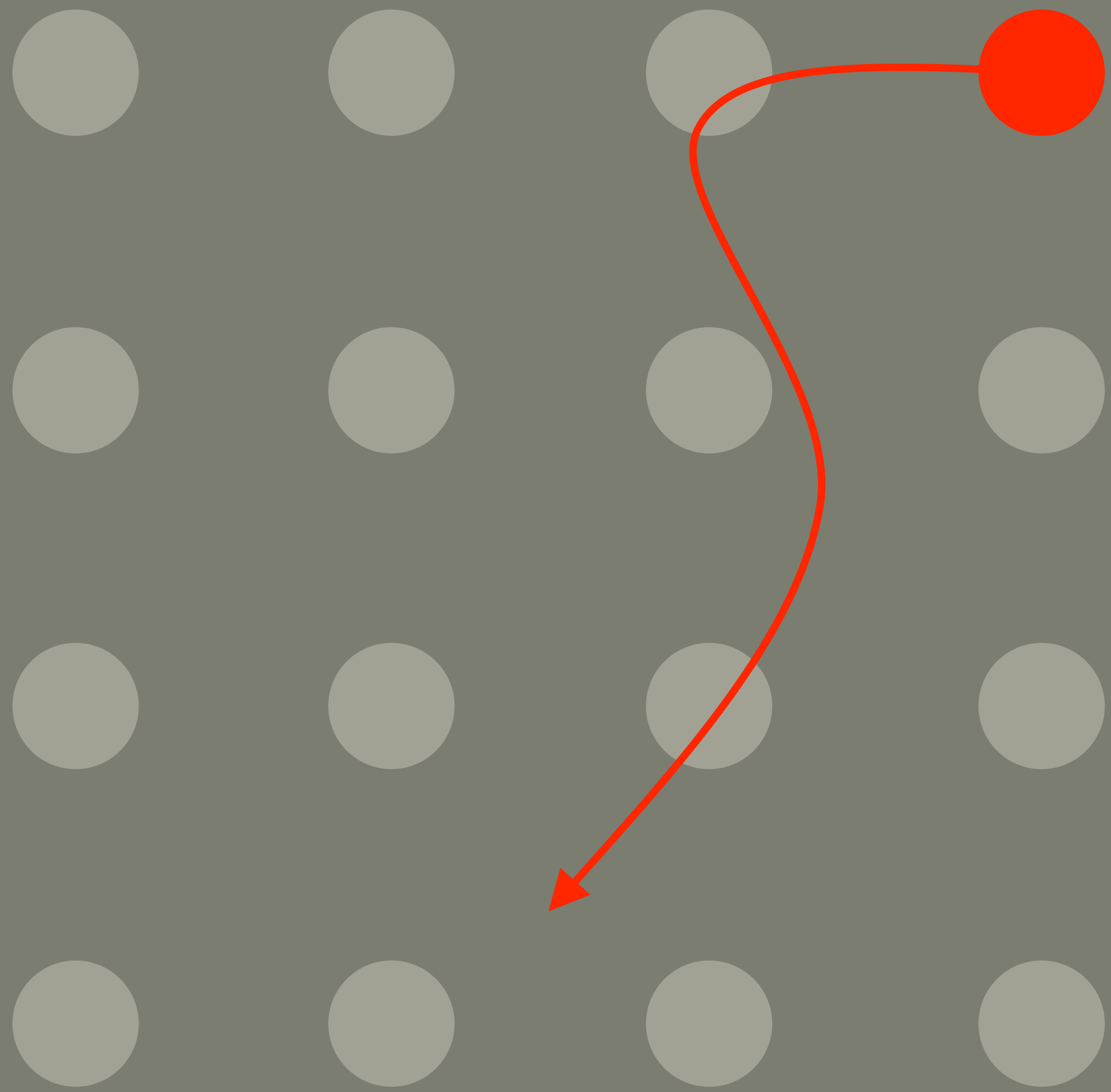
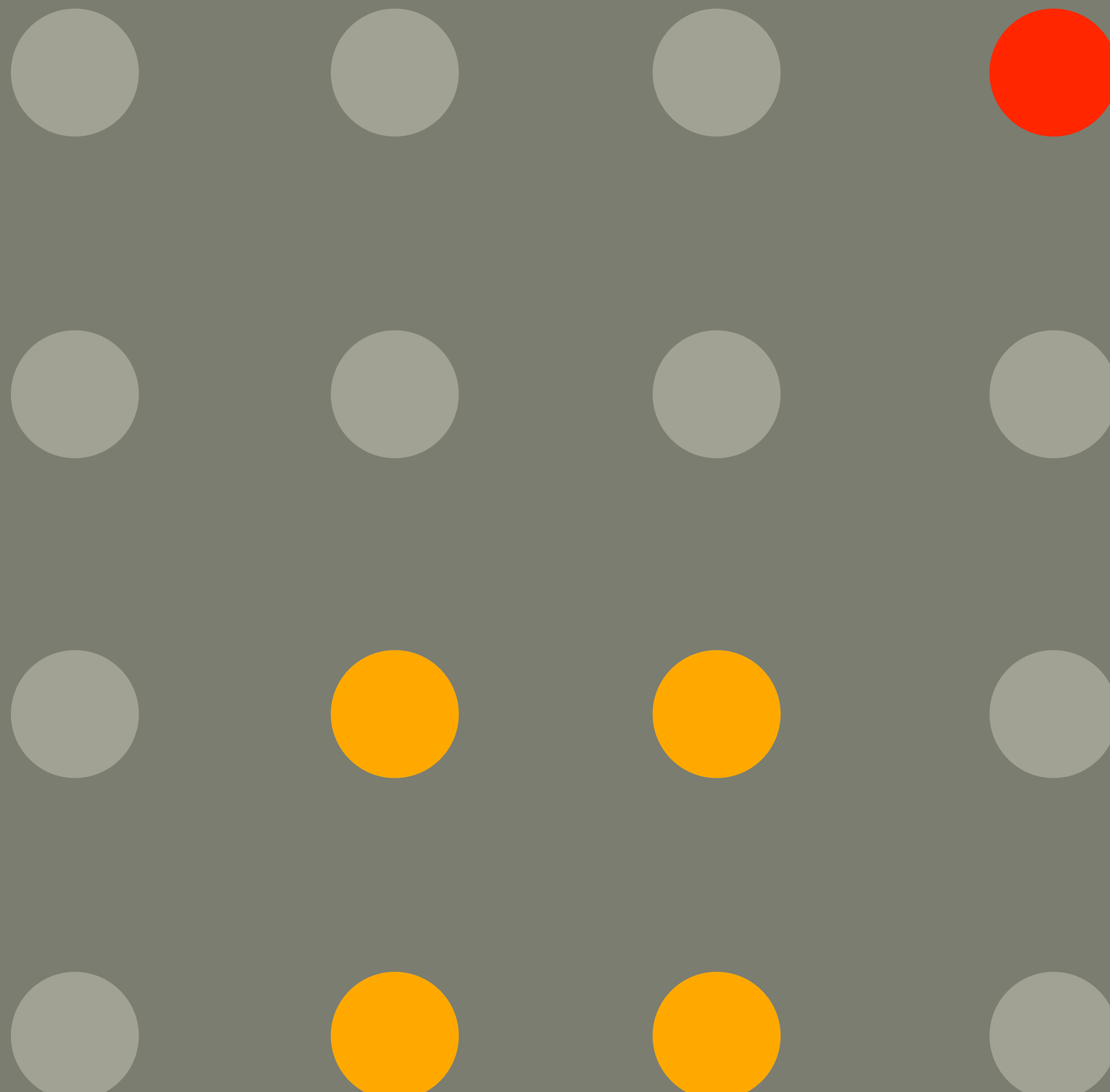
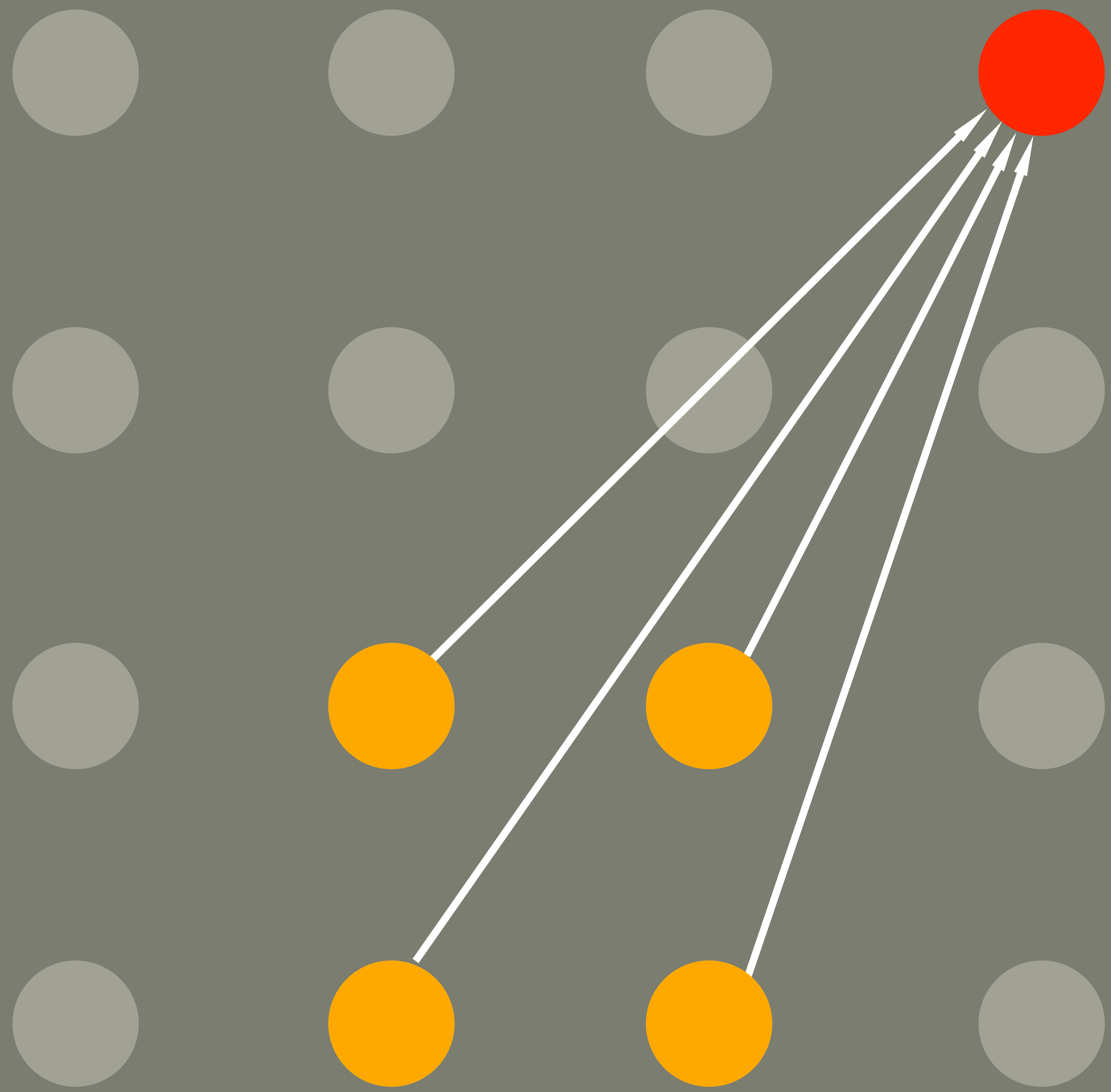


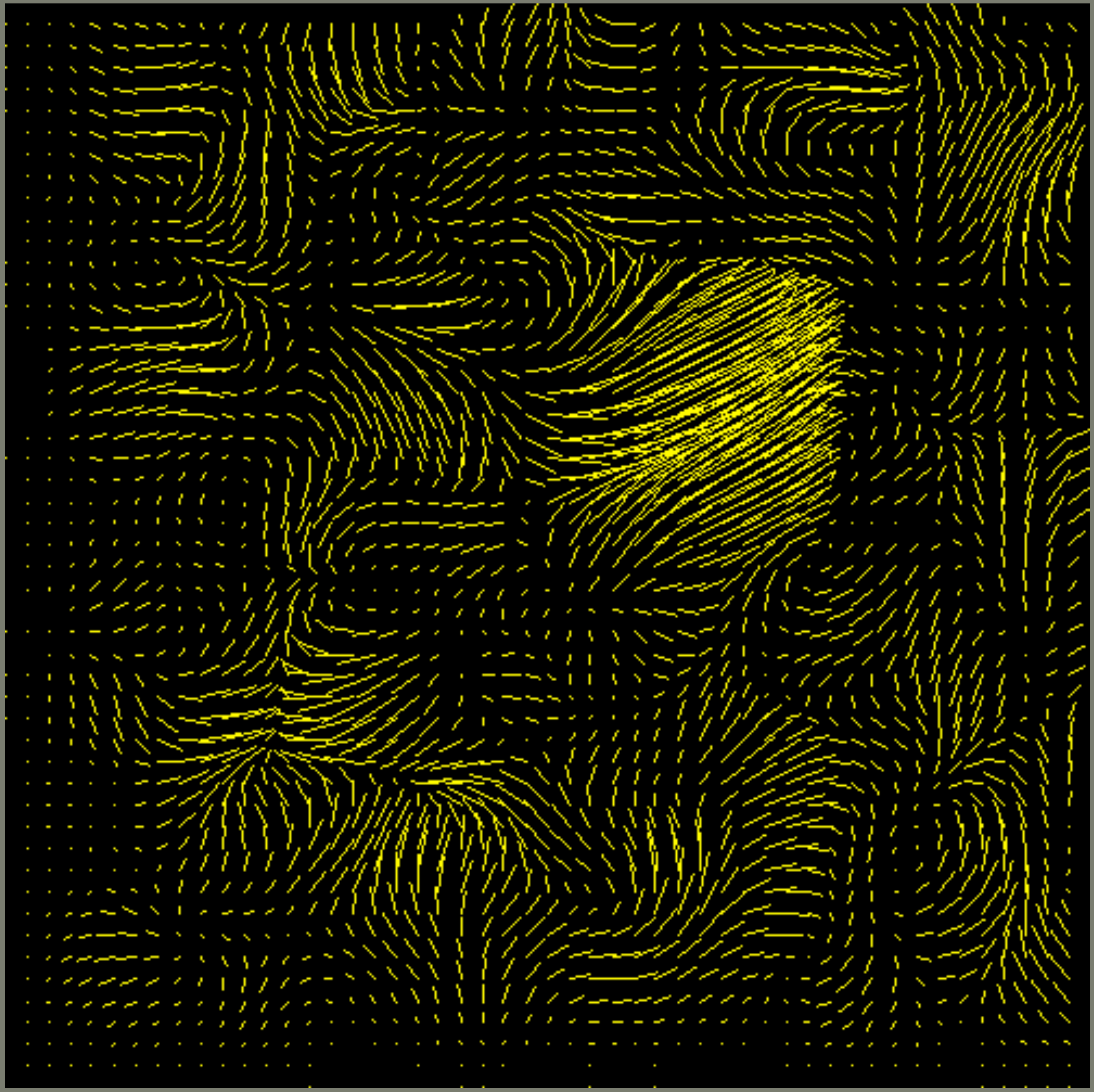
Advection



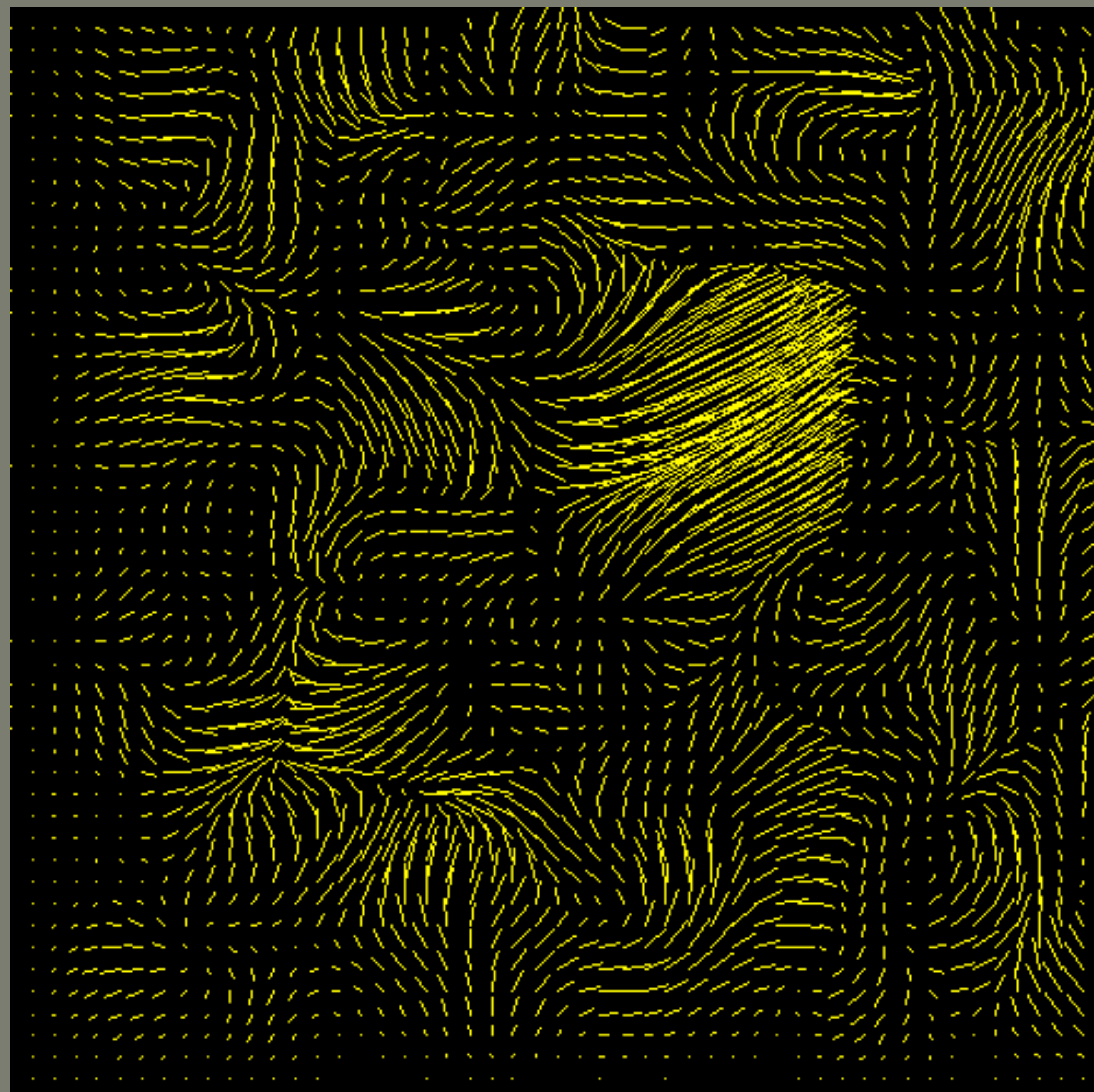
Advection



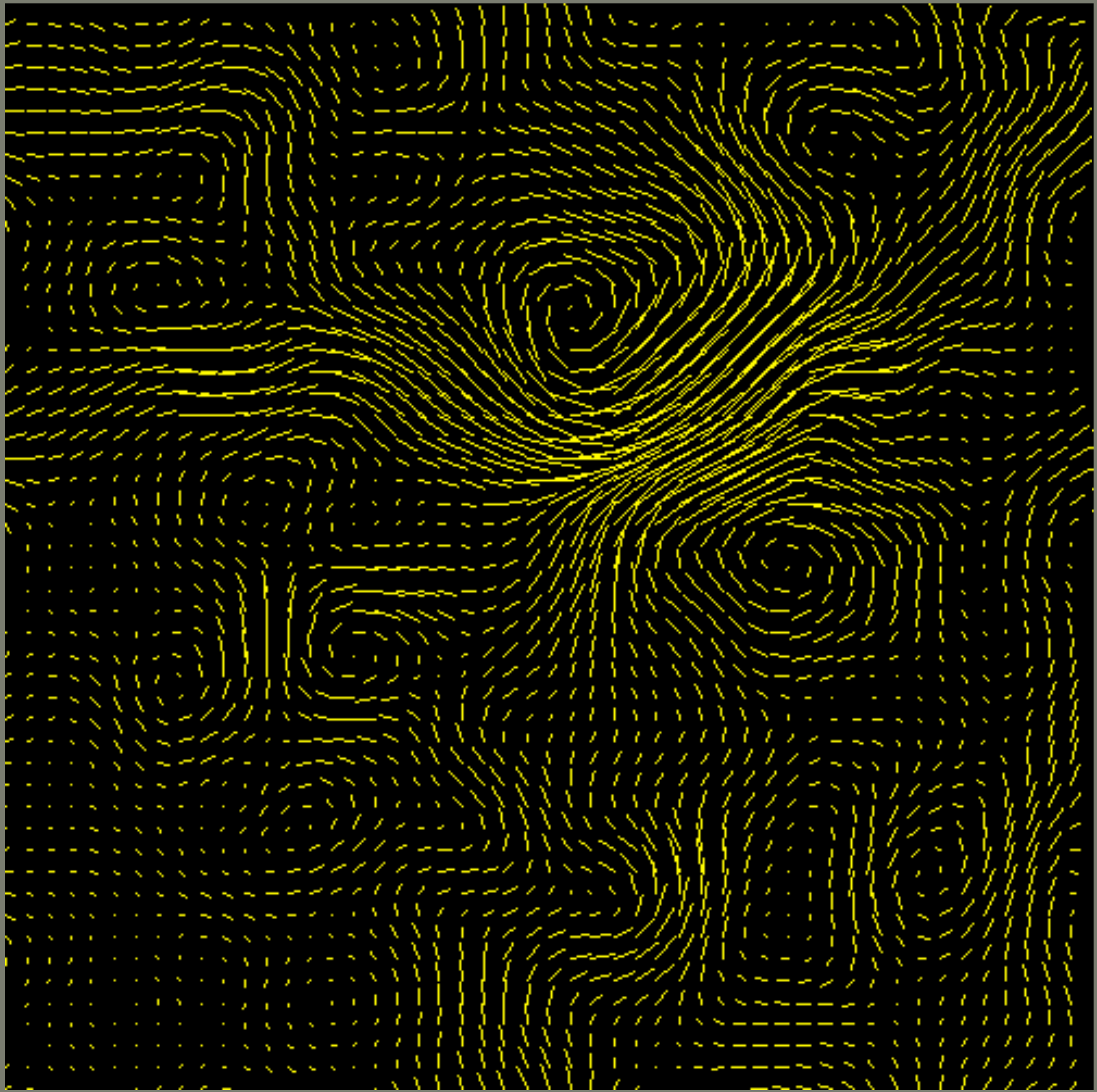
Projection



Projection

$P\Big($  $\Big)$

Projection



Diffusion

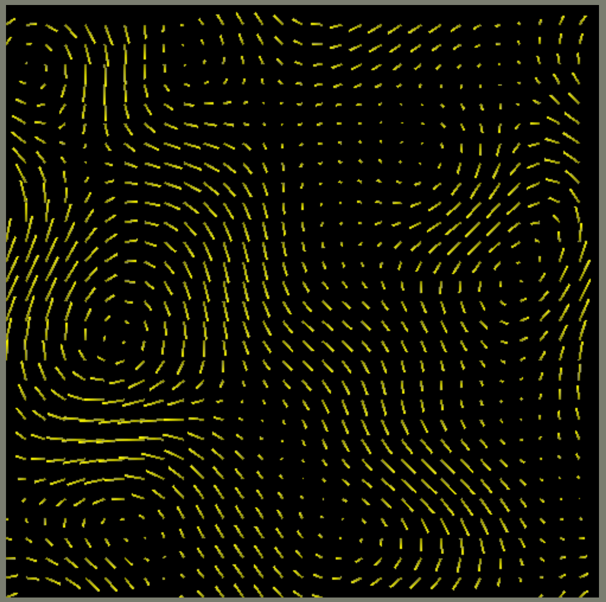
Diffusion

- Solved implicitly (like projection)

Diffusion

- Solved implicitly (like projection)
- I don't have a picture of this.

Add Forces (e.g. heat)



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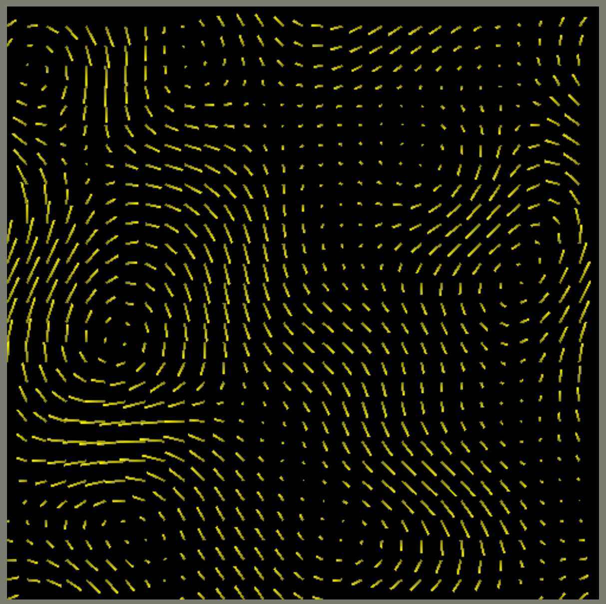

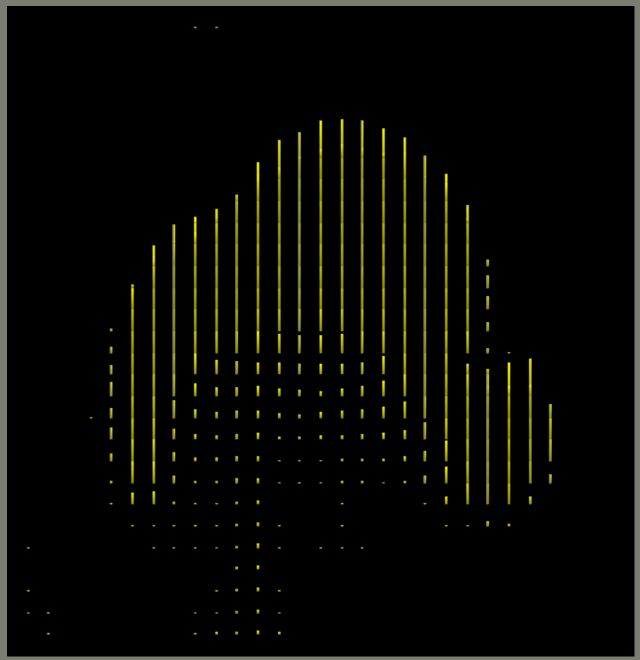

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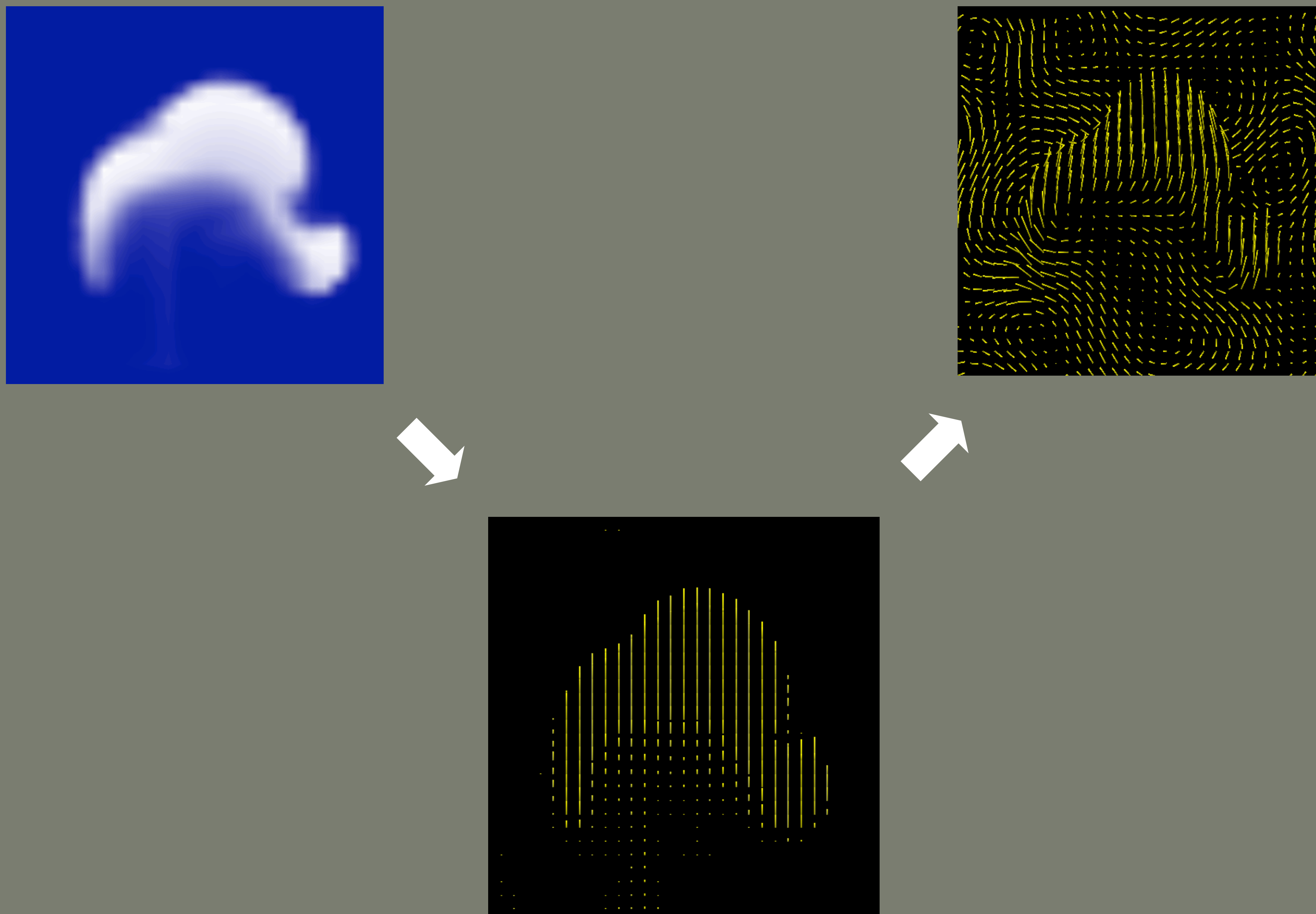

Simulation Recap

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- Decided Upon *grid-based* represenation.

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- Explicit Methods will not work.

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Simulation Recap

- Decided Upon *grid-based* represenation.
- Explicit Methods will not work.
- Stable Fluids solves all our problems…
  - …maybe.

Overview

- Last Week's Question: **Advection**
- Fluid Particles (recap)
- From Particles to PDEs
- The Navier Stokes Equations
- Stable Fluids Implementation
- Questions

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Questions

- Which phenomena does PDE method capture better? Why?
- Which phenomena does SPH capture better? Why?
- In the PDE implementation, how could we handle boundaries?
- How could we handle free surfaces?