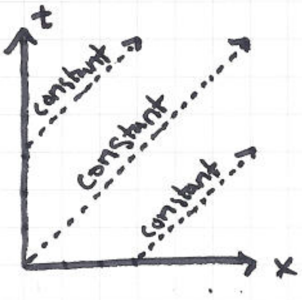
We are studying the very important case of a PDE that "pushes" (or "advects") a density field ρ according to a velocity field $\vec{u}$.

Case 1 ρ is a 1D function that moves to the right.

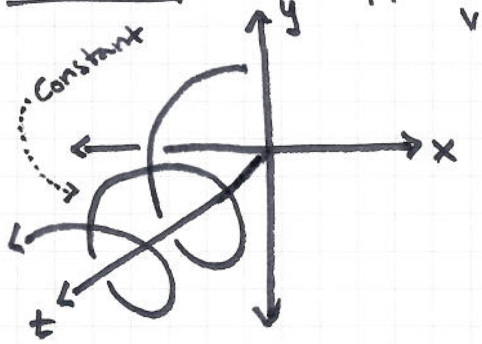


$$\begin{bmatrix} \frac{\partial \rho}{\partial t} & \frac{\partial \rho}{\partial x} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial \rho}{\partial x}$$

Case 2: We suppose that ρ is 2D & $\vec{u}$ defines a rotational velocity field.



$$\vec{u}(\vec{x}) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

By analogy with case 1:

$$\begin{bmatrix} \frac{\partial \rho}{\partial t}, & \frac{\partial \rho}{\partial x}, & \frac{\partial \rho}{\partial y} \end{bmatrix} \begin{bmatrix} u(x,y) \\ v(x,y) \\ 1 \end{bmatrix} = 0$$

$$\frac{\partial \rho}{\partial t} = - u(x,y) \frac{\partial \rho}{\partial x} - v(x,y) \frac{\partial \rho}{\partial y}$$

$$\frac{\partial \rho}{\partial t} = - \underbrace{\vec{u} \cdot \nabla}_{} \rho$$

The operator $u \cdot \nabla$ is called the advection operator.

Remark: Another way to look at it. Suppose we cover space with infinitely many particles with the following properties:

particle

ρ density

$\vec{x}$ position

$\frac{d\vec{x}}{dt}$ velocity.

$$\frac{d\rho}{dt} = 0$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x} \frac{dx}{dt} + \frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial \rho}{\partial x} \frac{dx}{dt} = - \underbrace{\vec{u} \cdot \nabla}_{} \rho$$

does this look familiar?